Review of Ch 10

Answer the following questions

1) Find the following limits:

a) $$\lim \frac{n^2 \ln n}{3^n} \left(\frac{\infty}{\infty}\right) \overset{L.R.}{=} \lim \frac{n^2\cdot\frac{1}{n} + 2n\ln n}{3^n \ln 3}$$

$$= \lim \frac{n+2n\ln n}{3^n \ln 3}\left(\frac{\infty}{\infty}\right) \overset{L.R.}{=} \lim_{n\to\infty} \frac{1+2n\cdot\frac{1}{n}+2\ln n}{3^n \ln^2 3}$$

$$= \lim_{n\to\infty}\frac{3+2\ln n}{3^n\ln^2 3}\left(\frac{\infty}{\infty}\right)\overset{L.R.}{=} \lim \frac{2/n}{3^n \ln^3 3}$$

$$= \lim_{n\to\infty} \frac{2}{n\, 3^n \ln^3 3} = 0$$

حل 2

ممكن استخدام نظرية Sandwich إذا لاحظنا ان $1 < \ln n < n^{ر}$ لأي ر > ٠ وذلك عندما n تكون كبيرة بشكل كافٍ

بأخذ ر = ١ فإن

$$\begin{array}{ccccc} \frac{n^2}{3^n} & < & \frac{n^2 \ln n}{3^n} & < & \frac{n^3}{3^n}\\ \downarrow & & \downarrow & & \downarrow \\ 0 & & & & 0\end{array}$$

as $n\to\infty$

by L.R. $\lim_{n\to\infty}\left.\frac{n^2}{3^n}\right.^{0} = 0$ and $\lim_{n\to\infty}\frac{n^3}{3^n}=0$.

so by Sandwich Thrm, $\lim \frac{n^2\ln n}{3^n} = 0$.

b) $\lim \left(3^n+5^n\right)^{\frac{1}{n}}$ (∞^0)

consider $\lim \ln\left(3^n+5^n\right)^{\frac{1}{n}} = \lim \frac{\ln(3^n+5^n)}{n} \left(\frac{\infty}{\infty}\right)$

$$\overset{L.R.}{=} \lim \left(\frac{3^n\ln 3+5^n\ln 5}{3^n+5^n}\right) = \lim \frac{\left(\frac{3}{5}\right)^n\ln 3+\ln 5}{\left(\frac{3}{5}\right)^n+1} = \boxed{\ln 5}$$

c) $\lim \dfrac{3^n\, 6^n}{2^{-n}\, n!} = \lim \dfrac{2^n\, 3^n\, 6^n}{n!} = \lim \dfrac{(36)^n}{n!} = 0$

2) Find the sum of the following Series:

a) $\sum_{n=0}^{\infty} e^{-2n}\left(2^n - 1\right) = \sum_{n=0}^{\infty}\left(\dfrac{2^n}{e^{2n}} - \dfrac{1}{e^{2n}}\right)$

$$= \sum_{n=0}^{\infty}\left(\left(\frac{2}{e^2}\right)^n - \left(\frac{1}{e^2}\right)^n\right)$$

$$\sum_{n=0}^{\infty}\left(\frac{2}{e^2}\right)^n = \frac{1}{1-\frac{2}{e^2}} \simeq 1.371 \quad \left(G.S.,\ \left|\frac{2}{e^2}\right| < 1\right)$$

$$\sum_{n=0}^{\infty}\left(\frac{1}{e^2}\right)^2 = \frac{1}{1-\frac{1}{e^2}} = 1.1565 \quad \left(G.S.,\ \left|\frac{1}{e^2}\right| < 1\right)$$

$$\therefore \sum_{n=0}^{\infty} e^{-2n}\left(2^n - 1\right) = 1.3711 - 1.1565 = \boxed{0.2146}$$

b) $\sum_{n=1}^{\infty}\left(\tan n - \tan(n-1)\right)$

Sol: The series is telescoping,

$S_1 = \tan 1 - \cancelto{0}{\tan 0}$, $S_2 = \left(\cancel{\tan 1} - \tan 0\right) + \left(\tan 2 - \cancel{\tan 1}\right) = \tan 2 - \cancelto{0}{\tan 0}$

$S_3 = \left(\cancel{\tan 1} - \tan 0\right) + \left(\cancel{\tan 2} - \cancel{\tan 1}\right) + \left(\tan 3 - \cancel{\tan 2}\right) = \tan 3 - \cancelto{0}{\tan 0}$

$\vdots$

$S_n = \left(\cancel{\tan 1} - \tan 0\right) + \left(\cancel{\tan 2} - \cancel{\tan 1}\right) + \dots + \left(\tan n - \cancel{\tan(n-1)}\right) = \tan n - \cancelto{0}{\tan 0}$

$\therefore \quad S_n = \tan n$

لاحظ أنه الدالة tan n تأخذ كل القيم على R على أي فترة طولها π / لذلك لا يمكن أن يكون
هناك رقم L في R يكون نهاية لـ tan n عندما n → ∞ لذا

$\lim S_n$ d.n.e. and hence the series diverges.

3) For what values of a, the G.S. converges and what is the sum

$$\sum_{n=0}^{\infty} (\sin a)^n = 1 + \sin a + \sin^2 a + \cdots$$

Sol: Clearly, the series is geometric series with $r = \sin a$, so it converges when $|\sin a| < 1$

or $a = (2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$ (كل المضاعفات الفردية ل $\frac{\pi}{2}$)

since $\sin((2n+1)\frac{\pi}{2}) = \mp 1$

and the series converges to $\frac{1}{1 - \sin a}$.

Determine whether the following series converge, converge abs., converge conditionally or div.?

4) $$\sum_{n=1}^{\infty} \frac{n + \sqrt{n} - 1}{n + n^3}$$

Sol: Take $a_n = \frac{n + \sqrt{n} - 1}{n + n^3}$, $b_n = \frac{n}{n^3} = \frac{1}{n^2}$,

$\sum b_n = \sum \frac{1}{n^2}$ converges (p-series, $p = 2 > 1$)

Consider $\lim \frac{a_n}{b_n} = \lim \left(\frac{n + \sqrt{n} - 1}{n + n^3}\right) \cdot n^2 = 1$

so by LCT, $\sum_{n=1}^{\infty} \frac{n + \sqrt{n} - 1}{n + n^3}$ converges.

5) $$\sum_{n=1}^{\infty} \frac{n+1}{(5n+2)\sqrt{n}}$$

Sol: Take $a_n = \frac{n+1}{(5n+2)\sqrt{n}}$, $b_n = \frac{n}{n\sqrt{n}} = \frac{1}{\sqrt{n}}$

$\sum b_n$ diverges (p-series, $p = \frac{1}{2} < 1$)

Consider $\lim \frac{a_n}{b_n} = \lim \frac{n^{3/2} + \sqrt{n}}{5n^{3/2} + 2\sqrt{n}} = \frac{1}{5}$

so by LCT, $\sum \frac{n+1}{(5n+2)\sqrt{n}}$ diverges.

6) $\sum_{n=0}^{\infty} \frac{2^n + n}{(n+1)!}$

Sol: Using ratio test:

$$\rho = \lim \frac{a_{n+1}}{a_n} = \lim \frac{2^{n+1} + (n+1)}{(n+2)!} \frac{(n+1)!}{2^n + n}$$

$$= \lim_{n\to\infty} \left(\frac{2 \cdot 2^n + n + 1}{2^n + n}\right) \cdot \left(\frac{1}{n+2}\right)$$

$$= \lim_{n\to\infty} \left(\frac{2 + (n+1)/2^n}{1 + \frac{n}{2^n}}\right) \cdot \frac{1}{n+2} = \left(\frac{2+0}{1+0}\right) \cdot 0 = 0$$

since $\rho < 1 \Rightarrow$ The series converges.

7) $\sum_{n=3}^{\infty} \frac{\left(\frac{1}{n}\right)}{\ln n \sqrt{\ln^2 n - 1}}$

Sol: Let $f(x) = \frac{1/x}{\ln x \sqrt{\ln^2 x - 1}}$,

1) Clearly f is cont. $\forall\ x \geq 3$.

2) $f(x) > 0 \quad \forall\ x \geq 3 \qquad (\ln^2 x > 1 \ \forall x \geq 3)$

3) يمكن اثبات تناقص الدالة بإثبات ان $f' < 0$ ، ولكن نظراً لصعوبة وطول حساب المشتقة يمكن اثبات التناقص اسهل باستخدام التعريف :

$x_1 < x_2 \Rightarrow \ln x_1 < \ln x_2$ - — - — . ①

Moreover, for $x > 3$, $\ln x > 1 \Rightarrow \ln^2 x_1 - 1 < \ln^2 x_2 - 1$

and hence $\sqrt{\ln^2 x_1 - 1} < \sqrt{\ln^2 x_2 - 1}$ --- ②

$$\therefore \left(x_1 \ln x_1 \sqrt{\ln^2 x_1 - 1}\right) < \left(x_2 \ln x_2 \sqrt{\ln^2 x_2 - 1}\right)$$

$$\frac{1}{\left(x_1 \ln x_1 \sqrt{\ln^2 x_1 - 1}\right)} > \frac{1}{\left(x_2 \ln x_2 \sqrt{\ln^2 x_2 - 1}\right)}$$

or $f(x_1) > f(x_2) \Longrightarrow f$ is ↘ .

Consider $\displaystyle\int_3^\infty \frac{1/x \; dx}{\ln x \sqrt{\ln^2 x - 1}} = \lim_{b \to \infty} \int_3^b \frac{1/x \; dx}{\ln x \sqrt{\ln^2 x - 1}}$

$$= \lim_{b \to \infty} \int_{\ln 3}^{\ln b} \frac{du}{u \sqrt{u^2 - 1}}$$

$u = \ln x$

$du = \frac{1}{x}\, dx$

$x = 3 \longrightarrow u = \ln 3$

$x = b \longrightarrow u = \ln b$

$$= \lim_{b \to \infty} \sec^{-1}|u| \Big]_{\ln 3}^{\ln b}$$

$$= \lim_{b \to \infty} \left[\sec^{-1}|\ln b| - \sec^{-1}|\ln 3|\right] = \frac{\pi}{2} - \sec^{-1}|\ln 3|$$

$\Longrightarrow$ by integral test $\displaystyle\sum_{n=3}^{\infty} \frac{1/n}{\ln n \sqrt{\ln^2 n - 1}}$ converges.

8) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(10 + e^{2n})}{2n}$

sol: Consider $\lim a_n = \lim \dfrac{\ln(10 + e^{2n})}{2n}$ $\left(\dfrac{\infty}{\infty}\right)$

$$\overset{L.R.}{=} \lim \left(\frac{\cancel{2} e^{2n}}{10 + e^{2n}}\right) \Big/ \cancel{2} \quad \left(\frac{\infty}{\infty}\right) \overset{L.R.}{=} \lim \frac{2e^{2n}}{2e^{2n}} = 1 \neq 0$$

So the series diverges by nth-term test.

9) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln \sqrt{n}}$

Sol: The series of abs. values $\sum_{n=2}^{\infty} \frac{1}{n \ln \sqrt{n}} = \sum \frac{2}{n \ln n}$

which is divergent by integral test (حلت سابقاً)

By AST, $u_n = \frac{2}{n \ln n}$

1) $u_n > 0 \quad \forall\, n \geq 2$

2) $u_n' < 0$ (أخرى) $\Longrightarrow u_n \searrow$.

3) $u_n \longrightarrow 0$

So it is convergent conditionally.

10) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n+1)}{n} \left(\frac{1}{2}\right)^n$

Sol: The series of abs values is $\sum_{n=1}^{\infty} \left(\frac{2n+1}{n}\right)\left(\frac{1}{2}\right)^n$

$\rho = \lim \frac{a_{n+1}}{a_n} = \lim \left(\frac{2n+3}{n+1}\right)\left(\frac{n}{2n+1}\right) \frac{2^n}{2^{n+1}} = 2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$

Since $\rho < 1 \Longrightarrow$ by ratio test, the series of abs. values converges and hence the original series

$\sum (-1)^{n+1} \left(\frac{2n+1}{n}\right)\left(\frac{1}{2}\right)^n$ converges abs.

11) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2 \sqrt{n}\,(3+\cos n)} < ??$

Sol: $\sqrt{n+1} < \sqrt{n+n} = \sqrt{2}\,\sqrt{n} \quad \forall\, n \geq 1,$

and $\cos n > -1 \Longrightarrow 3 + \cos n > 3 - 1 = 2$

$\therefore \frac{1}{3 + \cos n} < \frac{1}{2}$

$$\Rightarrow \quad \frac{\sqrt{n+1}}{n^2\sqrt{n}\,(3+\cos n)} < \frac{\sqrt{2}\ \cancel{\sqrt{n}}}{n^2\ \cancel{\sqrt{n}}\cdot 2} = \frac{1}{\sqrt{2}\ n^2}$$

since $\sum \frac{1}{\sqrt{2}\,n^2}$ converges (p-series $p=2>1$)

$\Rightarrow$ by DCT, $\sum \frac{\sqrt{n+1}}{n^2\sqrt{n}\,(3+\cos n)}$ converges.

12) Find the radius and the interval of conv. of the power series

$$\sum_{n=1}^{\infty} \frac{n!}{3.6.9.\dots 3n}\,(2x-1)^n$$

Sol: The series of abs. values $\sum_{n=1}^{\infty} \frac{n!}{3.6.9.\dots.3n}\,|2x-1|^n$

Using ratio test: $\rho = \lim \left| \frac{a_{n+1}}{a_n} \right|$

$$= \lim \frac{\overset{n+1}{\cancel{(n+1)!}}\ \ |2x-1|^{\cancel{n+1}}}{\cancel{3.6.9.\dots.3n}\,(3n+3)}\ \ \frac{\cancel{3.6.9.\dots.3n}}{\cancel{n!}\ \ \cancel{|2x-1|^n}}$$

$$= \lim \left(\frac{n+1}{3n+3}\right) |2x-1| = \tfrac{1}{3}\,|2x-1|.$$

which is convergent when $\rho < 1 \Rightarrow -3 < 2x-1 < 3$

$\Rightarrow -2 < 2x < 4 \Rightarrow -1 < x < 2$

so the radius of convergent

$$R = \frac{2-(-1)}{2} = \boxed{3/2}$$

ملحوظة هامة : لاحظ أن

$3.6.9.\dots.3n = 3^n(1.2.3.\dots.n) = 3^n n!$

وبالتالي يمكن إعادة كتابة المتسلسلة بالصيغة

$$\sum_{n=1}^{\infty} \frac{n!}{3^n\,n!}(2x-1)^n = \sum \left(\tfrac{1}{3}\right)^n (2x-1)^n$$

وهي متسلسلة هندسية بالتأكيد جربها عند الحدود / ويمكنك حلها

باستخدام طريقة المتسلسلات الهندسية، ولكن باستخدام الحل الأعلى يجب فحص الحدود.

At $x = -1$: The series is $\sum (-1)^n$ (بعد التبسيط) which is divergent.

At $x = 2$: The serie $\sum 1^n = \sum 1$ diverges so the interval of convergent is $(-1, 2)$