# 10.1 Sequences

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* A **sequence** is a list of numbers $a_1, a_2, \dots, a_n, \dots$
  - where $a_i$ are numbers with index $i$ "order"
  - it can be finite or infinite
  - it is a function that sends $1$ to $a_1$
    $2$ to $a_2$
    $\vdots$
    $n$ to $a_n$ "the $n^{th}$ term"

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**Exp** $a_n = \sqrt{n}$

$a_1 = 1$

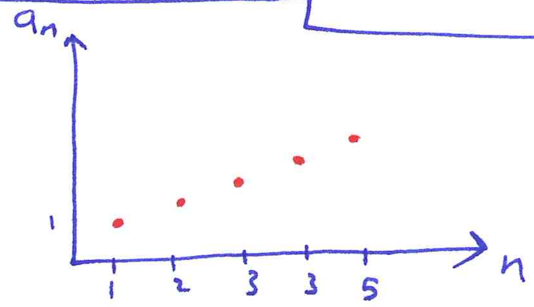
$a_2 = \sqrt{2}$

$a_3 = \sqrt{3}$

$\vdots$

$a_n = \sqrt{n}$

$\vdots$




$a_n \to \infty$ as $n \to \infty$

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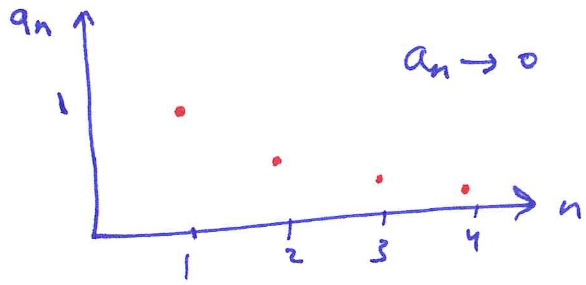
**Exp** $a_n = \frac{1}{n}$

$a_1 = 1$

$a_2 = \frac{1}{2}$

$a_3 = \frac{1}{3}$

$\vdots$




$a_n \to 0$ as $n \to \infty$

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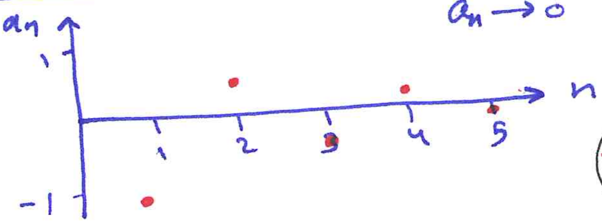
**Exp** $a_n = (-1)^n \frac{1}{n}$

$a_1 = -1$

$a_2 = +\frac{1}{2}$

$a_3 = -\frac{1}{3}$

$\vdots$




$a_n \to 0$ as $n \to \infty$

By Sandwich Th.

$-\frac{1}{n} \le (-1)^n \frac{1}{n} \le \frac{1}{n}$

$-\frac{1}{n} \to 0$, $\frac{1}{n} \to 0$

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**Def**: The sequence $\{a_n\}$ converges to the number $L$ "$a_n \to L$" $\equiv$ "$\lim_{n\to\infty} a_n = L$"

if for every number $\epsilon > 0$, there exists an integer $N$ s.t

for all $n > N \Rightarrow |a_n - L| < \epsilon$.

If such number $L$ does not exist, we say the sequence $\{a_n\}$ diverges. $\nearrow \infty$, $\searrow -\infty$

$\lim_{n\to\infty} a_n = \infty$ $\to$ diverges

$\lim a_n = -\infty$ $\to$ diverges

Exp ① Find $\lim_{n\to\infty} \frac{1}{n} = 0$

② Show that $\lim_{n\to\infty} \frac{1}{n} = 0$ Let $\epsilon > 0$, we need to show that there exist an integer $N$ such that for all

$$n > N \Rightarrow |a_n - L| < \epsilon$$

$$|\tfrac{1}{n} - 0| < \epsilon \Leftrightarrow |\tfrac{1}{n}| < \epsilon$$

This implication will hold if $\frac{1}{n} < \epsilon \Leftrightarrow n > \frac{1}{\epsilon}$

Take $N$ to be any integer greater than $\frac{1}{\epsilon}$

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③ Find $\lim_{n\to\infty} K = K$

④ Show that $\lim_{n\to\infty} K = K$ Let $\epsilon > 0$, we need to show that $\exists$ an integer $N$ s.t for all

$$n > N \Rightarrow |a_n - L| < \epsilon$$

$$|K - K| < \epsilon \Leftrightarrow 0 < \epsilon$$

$N$ can be any positive integer.

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**Th** Assume that $\{a_n\}$ and $\{b_n\}$ are sequences of real #'s, and let $A$ and $B$ be real #'s.

If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then

① $\lim_{n\to\infty} (a_n + b_n) = A + B$ ..... Sum Rule

② $\lim_{n\to\infty} (a_n - b_n) = A - B$ .... Difference Rule

③ $\lim_{n\to\infty} (K b_n) = K B$, ($K$ is any number) ... Constant Multiple Rule

④ $\lim_{n\to\infty} (a_n b_n) = A B$ ..... Product Rule

⑤ $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ "$B \neq 0$" .... Quotient Rule

**Exp** ① $\lim_{n\to\infty}\left(\frac{-\sqrt{3}}{n}\right) = -\sqrt{3}\lim_{n\to\infty}\frac{1}{n} = -\sqrt{3}\cdot 0 = 0$

② $\lim_{n\to\infty}\left(\frac{2n+5}{3n}\right) = \lim_{n\to\infty}\left(\frac{2}{3}+\frac{5}{n}\right) = \frac{2}{3}$

③ $\lim_{n\to\infty}\frac{n-2n^3}{n^3} = \lim_{n\to\infty}\left(\frac{1}{n^2}-2\right) = -2 + \lim_{n\to\infty}\frac{1}{n}\lim_{n\to\infty}\frac{1}{n}$

$= -2 + 0\cdot 0 = -2$

④ $\lim_{n\to\infty}\frac{3+\sqrt{8}n^5}{3+\sqrt{2}n^5} = \lim_{n\to\infty}\frac{\frac{3}{n^5}+\sqrt{8}}{\frac{3}{n^5}+\sqrt{2}} = \frac{0+\sqrt{8}}{0+\sqrt{2}} = 2$

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**Th** (Sandwich Th.)
Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers with $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$. If $a_n \le b_n \le c_n$ for all $n$ beyond some number $N$, then $\lim_{n\to\infty} b_n = L$.

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**Exp** ① $\lim_{n\to\infty}\frac{\sin n}{n} \to 0$ because $-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$

② $(-1)^n\frac{1}{n} \to 0$ because $-\frac{1}{n} \le (-1)^n\frac{1}{n} \le \frac{1}{n}$

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**Th 5** $\boxed{1}$ $\lim_{n\to\infty}\frac{\ln n}{n} = 0$ $\qquad \lim_{n\to\infty}\frac{\frac{1}{n}}{1} = \lim_{n\to\infty}\frac{1}{n} = 0$

$\boxed{2}$ $\lim_{n\to\infty}\sqrt[n]{n} = 1$ $\qquad \lim_{n\to\infty} e^{\ln n^{\frac{1}{n}}} = \lim_{n\to\infty} e^{\frac{\ln n}{n}} = e^0 = 1$

$\boxed{3}$ $\lim_{n\to\infty} x^{\frac{1}{n}} = 1 \quad (x>0)$ $\qquad \lim_{n\to\infty} e^{\ln x^{\frac{1}{n}}} = \lim_{n\to\infty} e^{\frac{\ln x}{n}} = e^0 = 1$

$\boxed{4}$ $\lim_{n\to\infty} x^n = 0 \quad (|x|<1)$ $\qquad \left(\frac{1}{2}\right)^n \to 0$ as $n\to\infty$ / $\left(-\frac{1}{2}\right)^n = (-1)^n\frac{1}{2^n} \to 0$ by Sandwich

$\to$ $x$ is fixed as $n\to\infty$

$\boxed{5}$ $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ $\qquad = \lim_{n\to\infty} e^{\ln\left(1+\frac{x}{n}\right)^n} = \lim_{n\to\infty} e^{\frac{\ln\left(1+\frac{x}{n}\right)}{\frac{1}{n}}} = \lim_{n\to\infty} e^{\frac{\frac{-x/n^2}{1+\frac{x}{n}}}{\frac{-1}{n^2}}} = e^x$

$\boxed{6}$ $\lim_{n\to\infty}\frac{x^n}{n!} = 0, \quad (\text{any } x)$ $\qquad \frac{\infty}{\infty} \Rightarrow \frac{nx^{n-1}}{(n!)'}, \frac{n(n-1)x^{n-2}}{(n!)''}, \dots, \frac{\text{constant}}{(n!)^{(n)}} = 0$

or $\cdot$ $0 \le \frac{xx\cdots x}{n(n-1)\cdots 1} \le \frac{x}{n} \to 0$ or $\frac{x}{n} \le \frac{xx\cdots x}{n(n-1)\cdots 1} \le 0$ Sandwich

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**Exp** ① $\lim_{n\to\infty} \frac{\ln n^3}{3n} = \lim_{n\to\infty} \frac{\ln n}{n} = 0$ $\boxed{1}$

② $\lim_{n\to\infty} \sqrt[n]{n^3} = \lim_{n\to\infty} n^{\frac{3}{n}} = \lim_{n\to\infty} \left(n^{\frac{1}{n}}\right)^3 = 1^3 = 1$ $\boxed{2}$

③ $\lim_{n\to\infty} \sqrt[n]{\pi n} = \lim_{n\to\infty} \pi^{\frac{1}{n}} \lim_{n\to\infty} n^{\frac{1}{n}} = (1)(1) = 1$
$\boxed{3}$ $\boxed{2}$

④ $\lim_{n\to\infty} \frac{\pi^{-n}}{e^{-n}} = \lim_{n\to\infty} \left(\frac{\pi}{e}\right)^{-n} = \lim \left(\frac{e}{\pi}\right)^n = 0$ $\boxed{4}$

⑤ $\lim_{n\to\infty} \left(\frac{n+1}{n-1}\right)^n = \lim_{n\to\infty} \left(\frac{n-1+2}{n-1}\right)^n = \lim_{n\to\infty} \left(1 + \frac{2}{n-1}\right)^n$

$u = n-1$

$1 + u = n$

$= \lim_{u\to\infty} \left(1 + \frac{2}{u}\right)^{u+1}$

$= \lim_{u\to\infty} \left(1 + \frac{2}{u}\right) \lim_{u\to\infty} \left(1 + \frac{2}{u}\right)^u$ $\boxed{5}$

$= (1)\,(e^2) = e^2$ (see the book) for another way.

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**Exp** Find a formula for the $n^{th}$ term of the sequence

① $1, -4, 9, -16, 25, \dots$ $a_n = (-1)^{n+1} \cdot n^2$, $n = 1, 2, 3, \dots$
$n \geq 1$

② $0, 3, 8, 15, 24, \dots$ $a_n = n^2 - 1$, $n \geq 1$

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**Exp** (Recursive Defined Sequence) Assume the following sequence converges, find it's limit.

$a_1 = 1$, $a_{n+1} = \frac{1}{2} a_n$

$a_2 = \frac{1}{2} a_1 = \frac{1}{2}$

$a_3 = \frac{1}{2} a_2 = \frac{1}{2}\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2$

$a_4 = \frac{1}{2} a_3 = \frac{1}{2}\left(\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^3$

$a_5 = \frac{1}{2} a_4 = \frac{1}{2}\left(\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^4$

$\vdots$

$a_n = \left(\frac{1}{2}\right)^{n-1}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{1}{2}\right)^{n-1}$

$= 2 \lim_{n\to\infty} \left(\frac{1}{2}\right)^n$

$= 2 \cdot 0$

$= 0$