$\boxed{10.10}$ The Binomial Series and Applications of Taylor Series

* The Binomial series of $f(x) = (1+x)^m$ is "using Taylor series expa."

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k \quad , \quad \underbrace{|x| < 1}_{\text{where the series converges absolutely.}} \quad , \quad m \text{ is constant}$$

where $\binom{m}{1} = m$, $\binom{m}{2} = \dfrac{m(m-1)}{2}$

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!} \quad \text{for } k \geq 3$$

Powers and Roots

Exp Find the first four terms of the binomial series for

$\boxed{1}$ $$(1+x)^{\frac{1}{2}} = 1 + \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} x^k = 1 + \frac{x}{2} + \frac{(\frac{1}{2})(-\frac{1}{2})\, x^2}{2} + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})\, x^3}{3!} + \cdots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots$$

$\boxed{2}$ $$\left(1 - \frac{x}{2}\right)^{-2} = 1 + \sum_{k=1}^{\infty} \binom{-2}{k} \left(\frac{-x}{2}\right)^k$$

$$= 1 + x + \frac{(-2)(-3)\left(\frac{-x}{2}\right)^2}{2} + \frac{(-2)(-3)(-4)\left(-\frac{x}{2}\right)^3}{3!} + \cdots$$

$$= 1 + x + \frac{3x^2}{4} + \frac{x^3}{2} + \cdots$$

Approximating Nonelementary Integrals

Exp Use series to estimate the following integrals with an error of magnitude less than 10^{-3}

$\boxed{1}$ $$\int_0^{0.1} \frac{dx}{\sqrt{1+x^4}} = \int_0^{0.1} (1+x^4)^{-\frac{1}{2}} dx = \int_0^{0.1} \left(1 + \sum_{k=1}^{\infty} \binom{-\frac{1}{2}}{k} x^{4k}\right) dx$$

$$= \int_0^{0.1} \left(1 - \frac{x^4}{2} + \frac{3x^8}{8} - \cdots\right) dx = \left[x - \frac{x^5}{10} + \frac{3x^9}{9(8)} - \cdots\right]_0^{0.1}$$

$$\approx x \Big|_0^{0.1} \approx 0.1 \quad \text{with error } |E| \leq \frac{(0.1)^5}{10} \approx 0.000001 .$$

$\boxed{2}$ $\displaystyle\int_0^{0.2} \frac{e^{-x}-1}{x}\,dx = \int_0^{0.2} \frac{1}{x}(e^{-x}-1)\,dx$ $\qquad e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$= \int_0^{0.2} \frac{1}{x}\left(\cancel{1} - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots \cancel{-1}\right) dx$$

$$= \int_0^{0.2} \left(-1 + \frac{x}{2!} - \frac{x^2}{3!} + \frac{x^3}{4!} - \cdots\right) dx$$

$$= \left[-x + \frac{x^2}{4} - \frac{x^3}{18} + \frac{x^4}{4(4!)} - \cdots\right]_0^{0.2} \approx -(0.2) + \frac{(0.2)^2}{4} - \frac{(0.2)^3}{18} \approx -0.19044$$

with error $|E| \leq \dfrac{(0.2)^4}{96} \approx 0.000002$

Arctangents: Remember that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

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with Leibniz's formula:

$$\frac{\pi}{4} = \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots + \frac{(-1)^n}{2n+1} + \cdots$$

- Note that $\tan^{-1} x = \displaystyle\int \frac{dx}{1+x^2}$

$$\Rightarrow \frac{1}{1+x^2} = \frac{d}{dx}\tan^{-1} x = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \frac{(-1)^{n+1} x^{2n+2}}{1+x^2}$$

- Integrate both sides from 0 to x $\Rightarrow$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \underbrace{\int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}\,dt}_{R_n(x)}$$

where $|R_n(x)| \leq \displaystyle\int_0^{|x|} t^{2n+2}\,dt = \frac{|x|^{2n+3}}{2n+3}$. $\to 0$ as $n \to \infty$ if $|x| \leq 1$

- If $|x| \leq 1$, then $\displaystyle\lim_{n\to\infty} R_n(x) = 0$ and so

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \quad |x| \leq 1.$$

Indeterminate forms

Exp Use series to evaluate the limits:

[1] $\lim_{x\to 1} \frac{\ln x}{x-1}$

Taylor Series of $\ln x$ about $x=1$ is

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \cdots$$

$$= \lim_{x\to 1}\left(1 - \frac{1}{2}(x-1) + \cdots\right) = 1$$

[2] $$\lim_{x\to 0} \frac{e^x-(1+x)}{x^2} = \lim_{x\to 0} \frac{1}{x^2}\left(e^x - 1 - x\right)$$

$$= \lim_{x\to 0}\frac{1}{x^2}\left(\cancel{1}+\cancel{x}+\frac{x^2}{2}+\frac{x^3}{3!}+\cdots\cancel{-1}\cancel{-x}\right)$$

$$= \lim_{x\to 0}\left(\frac{1}{2}+\frac{x}{3!}+\cdots\right) = \frac{1}{2}$$

[3] $$\lim_{y\to 0}\frac{y-\tan^{-1}y}{y^3} = \lim_{y\to 0}\frac{1}{y^3}\left[\cancel{y}-\left(\cancel{y}-\frac{y^3}{3}+\frac{y^5}{5}-\cdots\right)\right]$$

$$= \lim_{y\to 0}\left[\frac{1}{3}-\frac{y^2}{5}+\cdots\right] = \frac{1}{3}$$

Euler's formula

$i=\sqrt{-1}$, $i^2=-1$, $i^3=-i$, $i^4=1$, $i^5=i$

Recall that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \frac{i^6\theta^6}{6!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \rightarrow \text{Euler's formula}$$

θ is polar angle.

- Any complex number has the form $a+bi$, $a,b\in\mathbb{R}$.

Exp $e^{i\pi} = \cos\pi + i\sin\pi = -1$