10.2 Infinite Series

* An infinite series is the sum of an infinite sequence of numbers $a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n$

where
- a_n is the n^{th} term of the series
- $S_1 = a_1$ is the 1^{st} partial sum of the series
- $S_2 = a_1 + a_2$ is the 2^{nd} partial sum of the series

  ⋮
- $S_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$ is the n^{th} partial sum of series.
- If the sequence of partial sums converges to a limit L then we say the series converges and we write $a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L$
- If the sequence of partial sums of the serie does not converge, then we say the series diverges.

Exp $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$

Partial sums

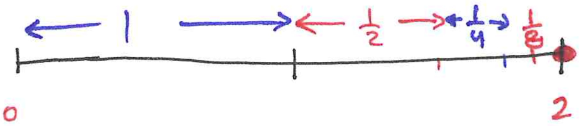
First $S_1 = 1 = 2 - 1$

second $S_2 = 1 + \frac{1}{2} = 2 - \frac{1}{2}$

Third $S_3 = 1 + \frac{1}{2} + \frac{1}{4} = 2 - \frac{1}{4}$

⋮

nth $S_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} = 2 - \frac{1}{2^{n-1}}$



- Note that this sequence of partial sums converges to 2 because $\lim_{n\to\infty} S_n = 2 - 0 = 2$.
- Thus, we say the sum of the infinite series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 2$

Geometric Series

Geometric series are series of the form:

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} a r^{n-1} \qquad *$$

where • a and r are fixed real numbers

• $a \neq 0$

• r is called the ratio and can be positive:

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots \qquad r = \frac{1}{2}, \quad a = 1$$

or negative:

$$1 - \frac{1}{3} + \frac{1}{9} - \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots \qquad r = \frac{-1}{3}, \quad a = 1$$

To determine the convergence and divergence of the geometric series $*$; we consider 3 cases:

① • If $r = 1$, then the nth partial sum of the geometric series is $S_n = a + a(1) + a(1)^2 + \cdots + a(1)^{n-1} = na$ and the series diverges because $\lim_{n \to \infty} S_n = \pm\infty$ depending on the sign of a.

② • If $r = -1$, then the nth partial sum of the series is $S_n = \cancel{a} - \cancel{a} + \cancel{a} - \cancel{a} + a + \cdots a(-1)^{n-1} = \begin{cases} 0 & \text{if } n \text{ even} \\ a & \text{if } n \text{ odd} \end{cases}$

Thus, the series diverges because the nth partial sum alternate between a and 0.

③ • If $r \neq 1$ and $r \neq -1$ (i.e $|r| \neq 1$) then we can determine the convergence or divergence as follows:

$$S_n = a + \cancel{ar} + \cancel{ar^2} + \cancel{\cdots} + \cancel{ar^{n-2}} + ar^{n-1}$$

$$rS_n = \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cdots + \cancel{ar^{n-1}} + ar^n$$

$$S_n - rS_n = a - ar^n \iff S_n(1-r) = a(1-r^n) \iff$$

$$S_n = \frac{a(1-r^n)}{1-r}, \quad r \neq 1$$

- If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$

$$\Rightarrow \lim_{n\to\infty} S_n = \frac{a}{1-r}.$$ Thus $\sum_{n=1}^{\infty} ar^{n-1}$ converges to $\frac{a}{1-r}$

- If $|r| > 1$, then $r^n \to \infty$ and the series diverges

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* If $|r| < 1$, the geometric series

$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $\frac{a}{1-r}$: $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$, $\underline{|r|<1}$

* If $|r| \geq 1$, the series diverges

Exp: $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = \frac{1}{1-\frac{1}{2}} = 2$

Exp: $1 - \frac{1}{3} + \frac{1}{9} - \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^{n-1} = \frac{1}{1 - -\frac{1}{3}} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$

Exp: Express the repeating decimal numbers as the ratio of two integers:

① $0.\overline{23} = 0.23\ 23\ 23 \cdots = \frac{23}{100} + \frac{23}{(100)^2} + \frac{23}{(100)^3} + \cdots = \sum_{n=1}^{\infty} \frac{23}{100}\left(\frac{1}{100}\right)^{n-1}$

$$= \frac{\frac{23}{100}}{1 - \frac{1}{100}} = \frac{\frac{23}{100}}{\frac{99}{100}} = \frac{23}{99}$$

② $0.\overline{7} = 0.777\cdots = \frac{7}{10} + \frac{7}{(10)^2} + \cdots$

$$= \frac{\frac{7}{10}}{1 - \frac{1}{10}} = \frac{\frac{7}{10}}{\frac{9}{10}} = \frac{7}{9}$$

③ $0.0\overline{6} = 0.0666\cdots = \frac{6}{100} + \frac{6}{1000} + \frac{6}{10000} + \cdots$

$$= \frac{\frac{6}{100}}{1 - \frac{1}{10}} = \frac{\frac{6}{100}}{\frac{9}{10}} = \frac{6}{90} = \frac{1}{15}$$

The n^{th} Term Test for Divergent Series

* If $\lim_{n\to\infty} a_n$ fails to exist or is different from zero, then $\sum_{n=1}^{\infty} a_n$ diverges

Exp ① The series $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$

② The series $\sum_{n=1}^{\infty} \sqrt{n}$ diverges becuse $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{n} = \infty$

③ The series $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\to\infty} (-1)^{n+1}$ does not exist

④ The series $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges because $\lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0$

Th 7 If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Note that Th7 does not say that if $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ converges : exp $\sum_{n=1}^{\infty} \frac{1}{n}$ "harmonic" series

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Th 8 If $\sum_{n=1}^{\infty} a_n = A$ and $\sum b_n = B$ are convergent series; then

1. Sum Rule : $\sum (a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. Difference Rule $\sum (a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. Constant Multiple Rule: $\sum k\, a_n = k \sum a_n = kA$, $k \in \mathbb{R}$

Note that ① Every nonzero constant multiple of divergent series is divergent

② If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum (a_n + b_n)$ and $\sum (a_n - b_n)$ both diverges.

"Telescoping Series"

Exp Find a formula for the n^{th} partial sum of the following series and use it to determine if the series converges or diverges. If the series converges find the sum.

① $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$

$$S_n = \left(1 - \cancel{\frac{1}{\sqrt{2}}}\right) + \left(\cancel{\frac{1}{\sqrt{2}}} - \cancel{\frac{1}{\sqrt{3}}}\right) + \left(\cancel{\frac{1}{\sqrt{3}}} - \cancel{\frac{1}{\sqrt{4}}}\right) + \cdots + \left(\cancel{\frac{1}{\sqrt{n-1}}} - \cancel{\frac{1}{\sqrt{n}}}\right) + \left(\cancel{\frac{1}{\sqrt{n}}} - \frac{1}{\sqrt{n+1}}\right)$$

$S_n = 1 - \frac{1}{\sqrt{n+1}}$ $\Rightarrow \lim_{n\to\infty} S_n = 1$. Thus, the series converges to 1 i.e $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right) = 1$

② $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ " we use partial fraction"

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}, \quad A = 1, \; B = -1$$

$$\sum_{n=1}^{n} \left[\frac{1}{n} - \frac{1}{n+1} \right] = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$S_n = 1 - \frac{1}{n+1}$$

$\lim_{n\to\infty} S_n = 1$. Thus, the series converges to 1.

i.e $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

Exp Find the sum of the following series

$$\sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n} \right) = (5+1) + \left(\frac{5}{2} + \frac{1}{3}\right) + \left(\frac{5}{4} + \frac{1}{9}\right) + \left(\frac{5}{8} + \frac{1}{27}\right) + \cdots$$

$$= \left[5 + \frac{5}{2} + \frac{5}{4} + \frac{5}{8} + \cdots\right] + \left[1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots\right]$$

$$= \frac{5}{1 - \frac{1}{2}} + \frac{1}{1 - \frac{1}{3}}$$

$$= \frac{5}{\frac{1}{2}} + \frac{1}{\frac{2}{3}}$$

$$= 10 + \frac{3}{2} = \frac{23}{2}$$
