10.3 The Integral Test

Corollary: A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges iff its partial sums (s_n) are bounded from above.

Exp $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^n + \cdots$ geometric series with $r = \frac{1}{2} < 1$

$$= \frac{1/2}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

Note that $s_n \le 1 \quad \forall\, n = 1, 2, 3, \ldots$

That is
$$s_1 = \frac{1}{2}$$
$$s_2 = \frac{1}{2} + \left(\frac{1}{2}\right)^2$$
$$s_3 = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3$$
$$\vdots$$
$$s_n = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^n.$$

note that since the serie converge $\Rightarrow a_n \to 0$ as $n \to \infty$

Exp $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ The harmonic series is divergent.

$$= 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> \frac{2}{4} = \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{> \frac{4}{8} = \frac{1}{2}} + \underbrace{\left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right)}_{> \frac{8}{16} = \frac{1}{2}} + \cdots$$

The sequence of the partial sums is not bounded above because we don't have $s_n \le s_{n+1}$.

- Thus, the harmonic series diverges to ∞. The process is very slow. That is after 178 million terms, its partial sum is 20.

Thm "The Integral Test"
Consider the series $\sum_{n=K}^{\infty} a_n$, where

- a_n is a sequence of postive terms
- $a_n = f(n)$ is s.t f is continuous, positive, decreasing on $[K, \infty)$.

Then the series $\sum_{n=K}^{\infty} a_n$ and the integral $\int_K^{\infty} f(x)\,dx$ both converges or both diverges.

Exp Does the following series converge / diverge?

① $\sum_{n=1}^{\infty} \frac{1}{n^2}$ $(a_n \to 0$ as $n \to \infty)$ so it may converge

$f(x) = \frac{1}{x^2}$ is continous, positive, decreasing function on $[1, \infty)$

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2} = \frac{1}{2-1} = \{1\}$$

" by exp.* " note that this is not sum we don't know the sum.

Thus, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the integral test.

② The p- series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$ } " by exp* "

③ $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ $(a_n \to 0$ as $n \to \infty)$ so it may converge

$f(x) = \frac{1}{x^2+1}$ is continous, positive, decreasing function on $[1, \infty)$

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2+1} = \lim_{b\to\infty} \tan^{-1} x \Big|_1^b$$

$$= \lim_{b\to\infty} \left[\tan^{-1} b - \tan^{-1} 1\right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Thus, the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges by the integral test.

④ $\sum_{n=1}^{\infty} n \sin(\frac{1}{n})$ diverges by the n^{th} term test

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = \lim_{x\to 0} \frac{\sin x}{x} = 1 \neq 0$$

⑤ $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ $(a_n \to 0$ as $n \to \infty)$ so it may converge

$f(x) = \frac{1}{2x-1}$ is continous, positive, decreasing function on $[1, \infty)$

$$\int_1^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_1^b \frac{dx}{2x-1} = \lim_{b\to\infty} \frac{1}{2} \ln|2x-1| \Big|_1^b = \lim_{b\to\infty} \frac{1}{2} \ln(2b-1) = \infty$$

Thus, the series diverges by the integral test.