

10.4 (Direct) Comparison Test

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Th 10 "Comparison Test"

- Let $\sum a_n, \sum c_n, \sum d_n$ be series with nonnegative terms.
- Suppose that $d_n \leq a_n \leq c_n$ for all $n > N$.

(a) If $\sum c_n$ converges, then $\sum a_n$ also converges.

(b) If $\sum d_n$ diverges, then $\sum a_n$ also diverges.

Exp Use the Comparison Test to determine if the following series converges or diverges?

① $\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2} \leq 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \rightarrow$ This is p-series with $p=2$ which converges

Thus, $\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2}$ converges by the (Direct) Comparison Test

② $\sum_{n=1}^{\infty} \frac{2n}{3n-1}$ diverges because $\lim_{n \rightarrow \infty} a_n \neq 0$
 $\lim_{n \rightarrow \infty} \frac{2n}{3n-1} = \frac{2}{3} \neq 0$

③ $\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}} \geq \sum_{n=1}^{\infty} \frac{1}{n} \rightarrow$ divergent series "harmonic series"

Thus $\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$ diverges by the comparison Test.
 $n+n+n > n+\sqrt{n}+0 \Leftrightarrow 3n > n+\sqrt{n} \Leftrightarrow n > \frac{n+\sqrt{n}}{3}$
 $\frac{1}{n} < \frac{3}{n+\sqrt{n}}$

④ $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n < \sum_{n=1}^{\infty} \left(\frac{n}{3n}\right)^n = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n \rightarrow$ geometric series which converges

Thus, $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$ converges by the Comparison Test

The Limit Comparison Test

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Th (Limit Comparison Test)

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$

1) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

2) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

3) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Exp Use the Limit Comparison Test to determine whether the following series converges or diverges:

① $\sum_{n=1}^{\infty} \frac{n-2}{n^3 - n^2 + 3}$ $a_n = \frac{n-2}{n^3 - n^2 + 3}$ $b_n = \frac{1}{n^2}$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{n-2}{n^3 - n^2 + 3}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{n^3 - 2n^2}{n^3 - n^2 + 3} = 1 > 0$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges "p-series with $p=2$ ".

Thus, $\sum_{n=1}^{\infty} \frac{n-2}{n^3 - n^2 + 3}$ converges by the Limit Comparison Test.

② $\sum_{n=1}^{\infty} \frac{1}{1 + \ln n}$ $a_n = \frac{1}{1 + \ln n}$ $b_n = \frac{1}{n}$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{1 + \ln n}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{1 + \ln n} = \lim_{n \rightarrow \infty} \frac{1}{\frac{1}{n}} = \infty$$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges "harmonic series".

Thus, $\sum_{n=1}^{\infty} \frac{1}{1 + \ln n}$ diverges by the Limit Comparison Test.

(3) $\sum_{n=1}^{\infty} \sin \frac{1}{n}$ $a_n = \sin \frac{1}{n}$ $b_n = \frac{1}{n}$

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$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 > 0$$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

Thus, $\sum_{n=1}^{\infty} \sin \frac{1}{n}$ diverges by limit Comparison Test.

(4) $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{3/2}}$ $a_n = \frac{(\ln n)^2}{n^{3/2}}$ $b_n = \frac{1}{n^{3/2}}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{\frac{(\ln n)^2}{n^{3/2}}}{\frac{1}{n^{3/2}}} = \lim_{n \rightarrow \infty} \frac{(\ln n)^2}{1} = \lim_{n \rightarrow \infty} \frac{2 \ln n}{3 n^2} \\ &= \lim_{n \rightarrow \infty} \frac{2}{3} \frac{\ln n}{n^3} = \frac{2}{3} \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{3 n^2} = \frac{2}{9} \lim_{n \rightarrow \infty} \frac{1}{n^3} \\ &= \frac{2}{9} (0) = 0 \end{aligned}$$

and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent p-series. Thus, $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{3/2}}$ converges by the limit Comparison Test.

EXP What about $\sum_{n=1}^{\infty} \frac{1-n}{n 2^n} = \sum_{n=1}^{\infty} \frac{1}{n 2^n} + \sum_{n=1}^{\infty} \frac{-1}{2^n}$

converges by the Direct Comparison Test

since $\frac{1}{n 2^n} < \frac{1}{2^n}$

converges since it is a geometric series

Thus, $\sum_{n=1}^{\infty} \frac{1-n}{n 2^n}$ converges since it is sum of two convergent series.