10.5 The Ratio and Root Tests

Th "Ratio Test"

Consider the infinite series $\sum a_n$ with positive terms.

Assume $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \rho$. Then

- if $\rho < 1$, then the series converges.
- if $\rho > 1$, then the series diverges. "or $\rho = \infty$"
- if $\rho = 1$, then the test is inconclusive.

Exp Apply Ratio test to

[1] $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{e^n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2}{e^{n+1}} \cdot \frac{e^n}{n^2}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 \frac{1}{e} = \frac{1}{e} < 1$$

Thus, the series converges by the ratio test.

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[2] $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!}$$

$$= \lim_{n\to\infty} \frac{n+1}{e} = \infty > 1$$

Thus, the series diverges by the ratio Test.

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[3] $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ "harmonic series which diverges"

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{n}{1} = 1$$ "Ratio Test is inclusive"

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ "p-series with p=2 whic converges"

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{(n+1)^2} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^2} = 1$$ "Ratio Test is inclusive"

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[4] $\displaystyle\sum_{n=1}^{\infty} \frac{(n+3)!}{3!\,n!\,3^n}$ converges by Ratio Test $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+4}{3n+3} = \frac{1}{3} < 1$ ✓

Th "The Root Test"

Consider the infinite Series $\sum a_n$ with $a_n \geq 0$ for $n \geq N$.

Assume $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$. Then

- if $\rho < 1$, then the series converges
- if $\rho > 1$ or infinit, then the series diverges
- if $\rho = 1$, then the test is inconclusive.

Exp Apply the root test to

[1] $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$ $\quad \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{n} - \frac{1}{n^2}\right)^n} = \lim_{n\to\infty}\left(\frac{1}{n} - \frac{1}{n^2}\right) = 0 < 1$

Thus, the series converges by the root test.

[2] $\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$ $\quad \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{\ln n}{n}\right)^n} = \lim_{n\to\infty} \frac{\ln n}{n} = 0 < 1$

Thus, the series converges by the root test.

[3] $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$ $\quad \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{3^n}{n^3}} = \lim_{n\to\infty} \frac{3}{(\sqrt[n]{n})^3} = \frac{3}{1^3} = 3 > 1$

Thus, the series diverges by the root test.

[4] $\sum_{n=1}^{\infty} \frac{1}{n}$ "harmonic series which diverges"

$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1$ "Root Test is inconclusive"

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ "p-series which converges $p=2$"

$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{(\sqrt[n]{n})^2} = \frac{1}{1^2} = 1$ " Ratio Test is inconclusive"

Exp Consider the recursive defined terms: $a_1 = 2$, $a_{n+1} = \frac{2}{n} a_n$. Does $\sum_{n=1}^{\infty} a_n$ converg

$a_2 = \frac{2^2}{1!}$, $a_3 = \frac{2^3}{2!}$, $a_4 = \frac{2^4}{3!}$, $a_5 = \frac{2^5}{4!}$, $a_6 = \frac{2^6}{5!}$ ... $a_n = \frac{2^n}{(n-1)!}$ $\boxed{n \geq 1}$

Apply Ratio Test $\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2}{n} = 0 < 1 \Rightarrow$ The series converges.