10.6 Alternating Series
Absolute and Conditional Convergence

* Alternating Series is a series in which the terms are alternately positive and negative.

Exp [1] Alternating harmonic series:

converge by Th follows

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots+\frac{(-1)^{n+1}}{n}+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$$

[2] Alternating geometric series:

converge geometric

$$-1+\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\cdots+\frac{(-1)^n}{2^n}+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n}=\frac{-1}{1+\frac{1}{2}}=\frac{-2}{3}$$

[3] Alternating series can diverge:

diverge by th n^{th} term test

$$1-2+3-4+5-6+\cdots+(-1)^{n+1}n+\cdots=\sum_{n=1}^{\infty}(-1)^{n+1}n$$

Th "The Alternating Series Test" "Leibniz's Test" Let $u_n=|a_n|$

The alternating Series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n=u_1-u_2+u_3-u_4+\cdots$ converges if

[1] $u_n>0$ for all n and

[2] $u_{n+1}\leq u_n$ for all $n\geq N$ "nonincreasing" and

[3] $\lim_{n\to\infty}u_n=0$

Exp The alternating harmonic series is then converges by the alternating Series Test. To prove [2] let $f(x)=\frac{1}{x}$. If $f'(x)\leq 0$ then f is nonicreasing $\forall x>0 \Leftrightarrow n=1,2,\cdots$

* If $u_n=\frac{10n}{n^2+16}$, for what values of n does the sequence satisfy [2].

Let $f(x)=\frac{10x}{x^2+16} \Rightarrow f'(x)=\frac{10(16-x^2)}{(x^2+16)^2}\leq 0$ whenever $x\geq 4$

Thus, $u_{n+1}\leq u_n$ for $n\geq 4$. That is, the sequence is nonincreasing for $n\geq 4$.

* Assume the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} U_n = U_1 - U_2 + U_3 - U_4 + \cdots$$ converge to L

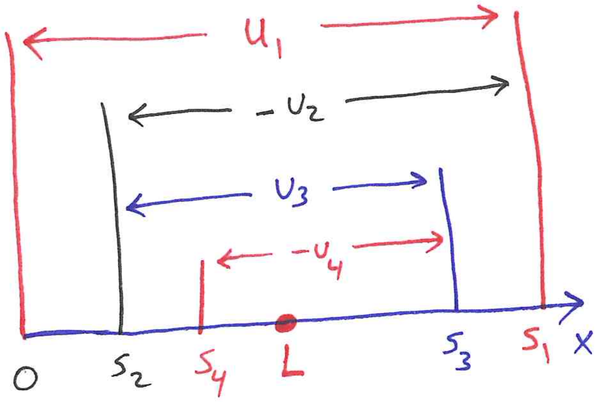
satisfies $\boxed{1}, \boxed{2}, \boxed{3}$

* $S_1 = U_1 > 0$

$S_2 = U_1 - U_2 = S_1 - U_2 > 0$

$S_3 = U_1 - U_2 + U_3 = S_2 + U_3 > 0$

$S_4 = U_1 - U_2 + U_3 - U_4 = S_3 - U_4$



* L lies between any two successive sums S_n and S_{n+1}.

$$S_n < L < S_{n+1}$$

* L differens from S_n by less than U_{n+1}:

$$\underbrace{|L - S_n|}_{\text{Absolute value of the error "magnitude" "Remainder"}} < U_{n+1} \quad \text{for } n \geq N$$

Exp ② Approximate the sum of $\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!}$ with an error of magnitude less than 5×10^{-6}.

First we find how many terms should be used to estimate the sum with an error $< 5 \times 10^{-6}$

$$\frac{1}{(2n)!} < 5 \times 10^{-6} \Rightarrow (2n)! > \frac{10^6}{5} = 200{,}000$$

$$\Rightarrow n \geq 5$$

$|\text{error}| < U_6 = |a_6| = \frac{1}{9!}$
$= 0.00000276$
$< 0.000005 = 5 \times 10^{-6}$

2nd $\quad S_5 = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} \approx 0.54030$

(*) goes here...

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① $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} + \frac{1}{256} - \cdots = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$

* Note that • $0.6640625 = S_8 < L = \frac{2}{3} < S_9 = 0.66796875$

• $\underbrace{L - S_n}_{\text{Remainder "Error"}} = L - S_8 = \frac{2}{3} - 0.6640625 = 0.0026041666$

$< U_9 = \frac{1}{256} = 0.00390625$

Def A series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges

Exp[1] $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$ converges absolutely because $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a convergent p-series

Exp[2] $\sum_{n=1}^{\infty} (-1)^{n+1} (0.1)^n$ converges absolutely because $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$ which is a convergent geometric series

Exp[3] $\sum_{n=1}^{\infty} (-1)^n \frac{\sin n}{n^2}$ converges absolutely by the Direct Comparasion Test because $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a convergent p-series.

Def: A series $\sum a_n$ is called converges conditionally if it converges but not absolutely.

Exp $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ "Alternating harmonic series" converges conditionally because $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$ which is the divergent harmonic serie

Exp $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converges by the Alternating Series Test

$\boxed{1} + \boxed{2}$ $\quad u_n = \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} = u_{n+1} > 0$

$\boxed{3}$ $\quad \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$

But $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ which is a divergent p-series

Thus, $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converges conditionally.

*** **Exp** $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$, $p > 0$ converges absolutely if $p > 1$

converges conditionally if $0 < p \leq 1$

Th If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

see Exp[1], Exp[2], Exp[3]

That is if the series converges absolutely, then it converges.

Proof: For each $n \Rightarrow -|a_n| \le a_n \le |a_n|$

$$\Rightarrow 0 \le a_n + |a_n| \le 2|a_n|$$

$$\Rightarrow 0 \le \sum_{n=1}^{\infty}\left(a_n + |a_n|\right) \le \underbrace{\boxed{2\sum_{n=1}^{\infty}|a_n|}}_{\text{converges}}$$

Thus, by the Direct Comparasion Test $\sum_{n=1}^{\infty}(a_n + |a_n|)$ converges

Since $a_n = a_n + |a_n| - |a_n|$

$$\Rightarrow \sum_{n=1}^{\infty} a_n = \underbrace{\boxed{\sum_{n=1}^{\infty}(a_n + |a_n|)}}_{\text{converge}} - \underbrace{\boxed{\sum_{n=1}^{\infty}|a_n|}}_{\text{converges}}$$

. Thus $\sum_{n=1}^{\infty} a_n$ converges

Examples ① $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3+n}{5+n}$ diverges by the n^{th} term test

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{3+n}{5+n} = 1 \neq 0$$

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② $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1+n}{n^2}$ converges conditionally since

[1]+[2] $f(x) = \frac{1+x}{x^2} = \frac{1}{x^2} + \frac{1}{x} \Rightarrow f'(x) = -\left(\frac{2}{x^3} + \frac{1}{x^2}\right) < 0 \Rightarrow$

$f(x)$ is decreasing and so $u_n > u_{n+1} > 0$

[3] $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1+n}{n^2} = \lim_{n\to\infty}\left(\frac{1}{n^2} + \frac{1}{n}\right) = 0$ so it converges

by the alternating series test.

But $\sum_{n=1}^{\infty}|a_n| = \underbrace{\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2}}}_{\text{convergent p-series}} + \underbrace{\boxed{\sum_{n=1}^{\infty}\frac{1}{n}}}_{\text{divergent harmonic series}}$ which diverges.

Th15 (The Alternating Series Estimation Theorem)

If $\sum_{n=1}^{\infty}(-1)^{n+1}u_n = L$, then $S_n = u_1 - u_2 + u_3 - \cdots + (-1)^{n+1}u_n$ approximate L with an error $|L - S_n| < u_{n+1} = |a_{n+1}|$. Furthermore, $S_n < L < S_{n+1}$ and the remainder $L - S_n$ has the same sign as a_{n+1}.