10.8 Taylor and Maclaurin Series

Def Let f be a smooth function "all derivatives exist" on an interval that contains the interior point a. Then

* The Taylor series generated by f at $x=a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

* The Maclaurin series generated by f is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Note that
- the Maclaurin series generated by f is the Taylor series generated by f at $x=0$.
- The function $f(x)$ could be approximated by the Talyor polynomials: $P_0(x), P_1(x), \cdots, P_n(x)$

$P_0(x) = f(a)$ — Polynomial of degree 0

$P_1(x) = f(a) + f'(a)(x-a)$ "linearization" Polynomial of degree 1

$P_2(x) = P_1(x) + \frac{f''(a)}{2!}(x-a)^2$ — Polynomial of degree 2

$P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$ — Polynomial of degree 3

$\vdots$

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Polynomial of degree $\underline{\underline{n}}$

Exp 1 Find the Talyor series of $f(x) = e^x$ at $x=0$.

$f(x) = e^x \Rightarrow f(0) = 1$

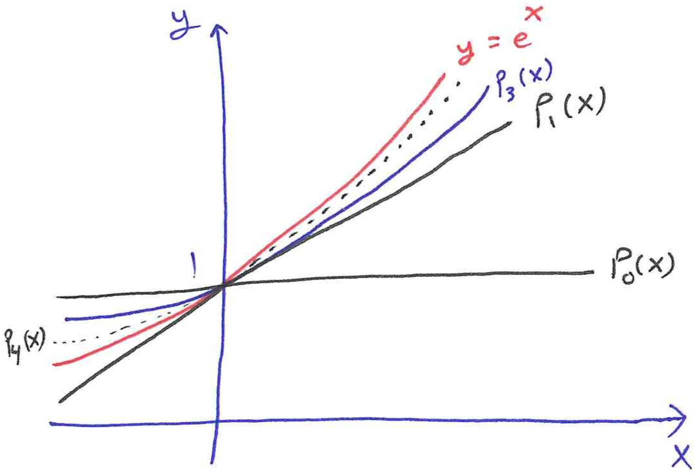
$f'(x) = e^x \Rightarrow f'(0) = 1$

$f''(x) = e^x \Rightarrow f''(0) = 1$

$f'''(x) = e^x \Rightarrow f'''(0) = 1$

$\vdots \qquad \vdots \qquad \vdots$

$f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = 1$



The Taylor series generated by f at $x=0$ is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots = \boxed{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

This is also The Maclaurin series of e^x.

The series converges to e^x for every x as $n \to \infty$

2 Find Taylor polynomials of order 0, 1, 2, 3

$P_0(x) = 1$

$P_1(x) = 1 + x$

$P_2(x) = 1 + x + \frac{x^2}{2!}$

$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

Exp Find the Talyor series and Taylor polynomials generated by $f(x) = \cos x$ at $x=0$

$f(x) = \cos x$ $\qquad$ $f'(x) = -\sin x$

$f''(x) = -\cos x$ $\qquad$ $f'''(x) = \sin x$

$f^{(4)}(x) = \cos x$ $\qquad$ $f^{(5)}(x) = -\sin x$

$\vdots \qquad \vdots$

$f^{(2n)}(x) = (-1)^n \cos x$ $\qquad$ $f^{(2n+1)}(x) = (-1)^{n+1} \sin x$

$f^{(2n)}(0) = (-1)^n$ $\qquad$ $f^{(2n+1)}(0) = 0$

The Talyor series generated by f at $x=0$ is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$= 1 + 0\cdot x - \frac{x^2}{2!} + 0\cdot x^3 + \frac{x^4}{4!} + \cdots$$

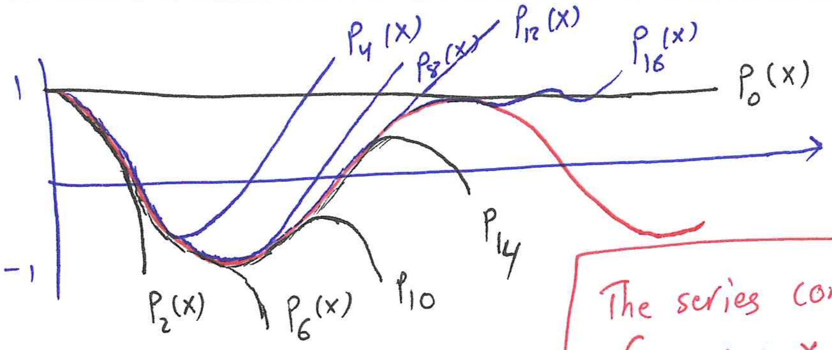
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$ $\boxed{\text{even function}}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

This is also the Maclaurin series of $\cos x$.

Talyor Polynomial of order $2n$ is

$$P_{2n}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}$$



$P_0(x) = 1$

$P_2(x) = 1 - \frac{x^2}{2!}$

$\boxed{\text{The series converges to } \cos x \text{ for every } x \text{ as } n \to \infty}$

Exp Find Taylor series generated by $f(x) = \sin x$ at $x = 0$.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

converges for every x to $\sin x$ as $n \to \infty$.

Exp Find the Maclaurin series for the function $\cosh x$.

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left[e^x + e^{-x}\right]$$

$$= \frac{1}{2}\left[1 + \cancel{x} + \frac{x^2}{2!} + \cancel{\frac{x^3}{3!}} + \cdots + 1 - \cancel{x} + \frac{x^2}{2!} - \cancel{\frac{x^3}{3!}} + \cdots\right]$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

Exp Find the Maclaurin series for the function $f(x) = \begin{cases} 0, & x = 0 \\ e^{-1/x^2}, & x \neq 0 \end{cases}$

The Maclaurin series of $f(x)$ is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$= 0 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + \cdots$$

$$= 0 + 0 + 0 + 0 + 0 + \cdots$$

$$= 0$$

The series converges for every x but converges to $f(x)$ only at $x = 0$

Thus, the Taylor series generated by $f(x)$ does not converge to $f(x)$ as $n \to \infty$.