10.9 Convergence of Taylor Series

Th (Taylor's Theorem)

Assume $f, f', f'', \dots, f^{(n)}$ are continuous on $[a,b]$ and $f^{(n)}$ is differentiable on (a,b), then there exists a number $c \in (a,b)$ such that:

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

* Note that Taylor's Theorem is a generalization of the MVT.

* If we change b by x, we get Taylor's formula:

$$f(x) = \underbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{P_n(x)} + R_n(x)$$

where

- $R_n(x)$ is the Remainder of order n "or the error term" results from approximating f by $P_n(x)$ on an open interval I contains a given by

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{where } c \in (a,x)$$

* Convergence of Taylor Series:

If $\lim_{n\to\infty} R_n(x) = 0$ for all $x \in I$, then the Taylor Series generated by f at $x=a$ converges to f on I:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

78

Exp show that the Taylor series generated by $f(x) = e^x$ at $x=0$ converges to $f(x)$ for every real value x.

$f(x) = e^x$ is smooth on $\mathbb{R} = (-\infty, \infty)$. Thus, has all orders of all derivatives. Taylor's formula is

$c \in (0,x)$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + R_n(x), \text{ where}$$

$$R_n(x) = \frac{e^c}{(n+1)!} x^{n+1}. \text{ Note that } \lim_{n\to\infty} R_n(x) = e^c \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$$

Thus, the series converges to e^x for every $x \Rightarrow$ Th 5 Sec 10.1

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

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* Note that $e = \sum_{k=0}^{\infty} \frac{1}{k!}$ as a series: $c \in (0,1)$, $0 < c < 1$, $e^0 < e^c < e < 3$

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n(1), \text{ where } R_n(1) = \frac{e^c}{(n+1)!} < \frac{3}{(n+1)!}$$

Th (The Remainder Estimation Theorem)

Given Taylor's formula: $f(x) = P_n(x) + R_n(x)$.

If $|f^{(n+1)}(t)| \le M$ for all $t \in (a,x)$, then the Remainder

$$\boxed{|R_n(x)| \le \frac{M}{(n+1)!} |x-a|^{n+1}.} \ *$$

Further, if $*$ holds for every n, then the series converges to $f(x)$.

Exp show that the Taylor series for $\sin x$ at $x=0$ converges for all x.

show that the Taylor series for $\sin x$ at $x=0$ is

$c \in (a,x) = (0,x)$

Recall that Taylor's formula for $\sin x$ at $x=0$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + R_{2n+1}(x).$$

Now apply The Remainder Estimation Theorem:

$$|R_{2n+1}(x)| = \left| \frac{f(c)}{(2n+2)!} x^{2n+2} \right| \le \frac{|x|^{2n+2}}{(2n+2)!}.$$

But $\lim_{n\to\infty} |R_{2n+1}(x)| \le \lim_{n\to\infty} \frac{|x|^{2n+2}}{(2n+2)!} = 0 \quad \forall x$

$\Rightarrow$ The Maclaurin series for $\sin x$ converges to $\sin x \ \forall x$.

$\Rightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$.

Exp Show that Taylor series for $\cos x$ at $x=0$ converges to $\cos x$ for every value of x.

The Taylor's formula for $\cos x$ at $x=0$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + R_{2n}(x).$$

Since $|\cos x|$ and $|$it's all derivative$| < 1$ we apply the Remainder Estimation Theorem with $\underline{M=1}$

$$|R_{2n}(x)| \le \frac{|x|^{2n+1}}{(2n+1)!}.$$ Now for every x we have $R_{2n}(x) \to 0$ as $n \to \infty$ by Th5 sechi 10.1

Therefore, the series converges to $\cos x$ for every x. Thus,

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Exp Find the first four nonzero terms in the Maclaurin series for the functions:

1. $$\frac{1}{3}(2x + x\cos x) = \frac{2x}{3} + \frac{x}{3}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$$
$$= \frac{2x}{3} + \frac{x}{3} - \frac{x^3}{2!\,3} + \frac{x^5}{3(4!)} - \frac{x^7}{3(6!)} + \cdots$$
$$= x - \frac{x^3}{6} + \frac{x^5}{72} - \frac{x^7}{2160} + \cdots$$

2. $$e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$
$$= \left(\underline{\underline{1 + x}} + \cancel{\frac{x^2}{2!}} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - \left(\cancel{\frac{x^2}{2!}} + \frac{x^3}{2!} + \frac{x^4}{2!\,2!} + \frac{x^5}{2!\,3!} + \cdots\right)$$
$$+ \left(\frac{x^4}{4!} + \frac{x^5}{4!} + \frac{x^6}{2!\,4!} + \cdots\right)$$
$$= \underline{\underline{1 + x}} - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

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3 $\cos 2x$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!} = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots$$

$$= 1 - \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} - \frac{2^6 x^6}{6!} + \cdots$$

Exp For what values of x can we replace $\sin x$ by $x - \frac{x^3}{3!}$ with an error of magnitude no more than 3×10^{-4}?

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

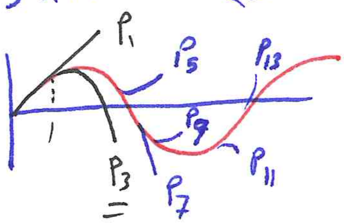
Using the Alternating Series Estimation Theorem:

The error after $\frac{x^3}{3!} < \left| \frac{x^5}{5!} \right|$. Therefore the error

$u_{n+1} = |a_{n+1}|$

will be less than 3×10^{-4} $\iff$ $\frac{|x|^5}{120} < 3 \times 10^{-4}$ $\iff$

$$|x| < \sqrt[5]{360 \times 10^{-4}} \simeq 0.514.$$



Exp Estimate the error if $P_3(x) = x - \frac{x^3}{6}$ is used to estimate the value of $\sin x$ at $x = 0.1$

$f(x) = \sin x = P_3(x) + R_3(x)$, where $R_3(x) = \frac{f^{(4)}(c)}{4!}(x-0)^4 = \frac{f^{(4)}(c)}{4!} x^4$

Apply the Remainder Estimation Th. with $M = 1$

$$\text{Error} = |R_3(x)| \le \frac{|x^4|}{4!} = \frac{(0.1)^4}{24} < 4.2 \times 10^{-6}$$

$M \le |f^{(4)}(c)| \le 1$