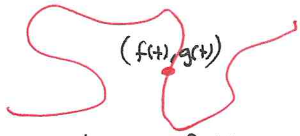
11.1 Parametrization of Plane Curves

* We may describe the movement of a particle in the xy plane at position t by $(x(t), y(t)) = (f(t), g(t))$



position of the particle at time t
not a function

Def If x and y are given as functions

$$x = f(t) \text{ and } y = g(t) \quad , \; t \in I,$$

then the set of points $(x, y) = (f(t), g(t))$ is a parameteric curve.

Note that
1. $x = f(t)$ and $y = g(t)$ are called parametric equations.
2. the variable t is called the parameter of the curve.
3. the interval I is called the parameter interval.

   $\Rightarrow$ If $I = [a, b]$ closed interval, then the point $(f(a), g(a))$ is the initial point and the point $(f(b), g(b))$ is the terminal point.
4. We say that we have parametrized the curve, if we find 1 and 3. That is 1 and 3 give a parametrization of the curve.

Exp Given the parametric equation and parameter interval:

$$x = t^2 \quad , \quad y = t + 1 \quad , \quad -\infty < t < \infty$$

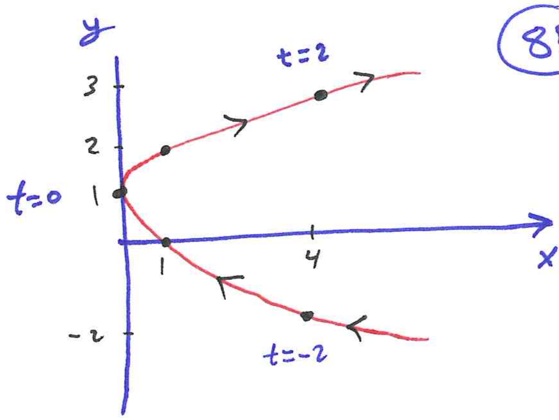
1. Find the Cartesian ($\to$ algebraic) equation by eliminating the parameter t
2. Idenitify the particle's path by sketching the cartesian equation
3. Find the direction of motion

1. Cartesian equation: $x = t^2 = (y-1)^2 \Leftrightarrow x = (y-1)^2$

   Note that sometimes it's difficult or even impossible to eliminate the parameter t.

The curve that represents the particle movement.

t	-3	-2	-1	0	1	2	3
x	9	4	1	0	1	4	9
y	-2	-1	0	1	2	3	4

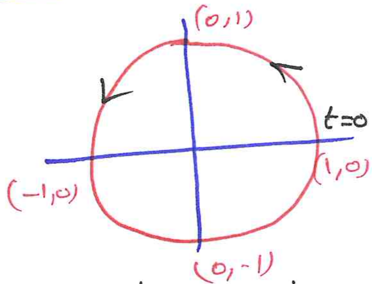


Exp Graph the parametric curve of $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$

- We can eliminate the parameter t by:

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1 \Leftrightarrow \boxed{x^2 + y^2 = 1}$$ cartesian equation

- Initial point is $(\cos 0, \sin 0) = (1, 0)$
- Terminal point is $(\cos 2\pi, \sin 2\pi) = (1, 0)$
- $t = \pi \Rightarrow$ the position is $(-1, 0)$



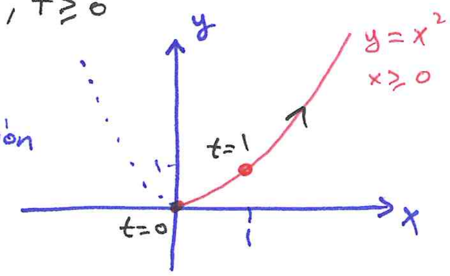
Direction: counterclockwise

Exp Graph the particle's movement and direction if its parametric equation and parameter interval is [1] $x = \sqrt{t}$, $y = t$, $t \ge 0$

- We can eliminate the parameter t

$$y = t = x^2 \Leftrightarrow \boxed{y = x^2}$$ Cartesian equation

- Initial point is $(\sqrt{0}, 0) = (0, 0)$
- No terminal point
- $t = 1 \Rightarrow$ the position is $(1, 1)$



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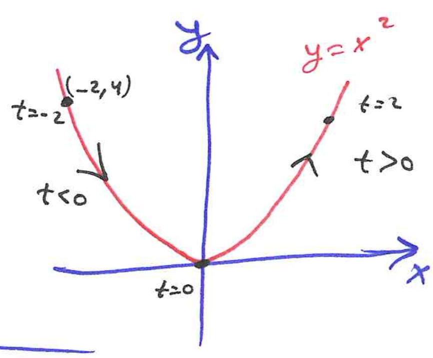
[2] $x = t$, $y = t^2$, $-\infty < t < \infty$

- We can eliminate the parameter t

$$y = t^2 = x^2 \Leftrightarrow \boxed{y = x^2}$$ → Cartesian equation

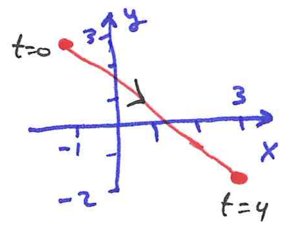
- no initial point
- no terminal point



Exp Find a parameterization for the line passes throw the points (a,b) and (c,d).

- A Cartesian equation is $y-b=m(x-a)$ where the slope $m=\dfrac{d-b}{c-a}$, $c\neq a$
- Set the parameter $t=x-a$
- Hence, $x=a+t$, $y=b+mt$, $-\infty<t<\infty$ parameterizes the line.

The line segment with endpoints $(-1,3)$ and $(3,-2)$ $\quad m=\dfrac{-2-3}{3+1}=\dfrac{-5}{4}$



$$\left\{x=-1+t,\ y=3-\frac{5}{4}t,\ 0\le t\le 4\right.$$

$$\{x=-1+4t,\ y=3-5t,\ 0\le t\le 1\}$$

Both parametrization give the same segment

Exp sketch and identify the path by the point $P(x,y)$ if

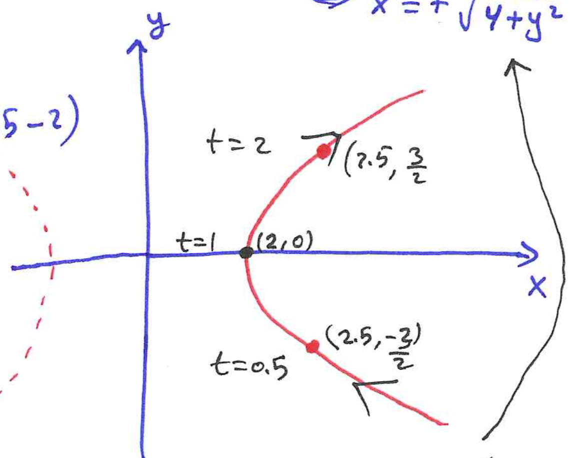
$$x=t+\frac{1}{t},\quad y=t-\frac{1}{t},\quad t>0$$

- We can eliminate the parameter t by:

$$\left.\begin{aligned}x+y&=2t\\ x-y&=\frac{2}{t}\end{aligned}\right\}\to (x+y)(x-y)=4 \iff x^2-y^2=4 \iff x^2=4+y^2$$

$$\iff x=+\sqrt{4+y^2}$$

- at $t=0.5 \Rightarrow$ the position is $(0.5+2,\ 0.5-2)$
  $\Rightarrow (2.5,-\frac{3}{2})$
- at $t=2 \Rightarrow$ the position is $(2.5,\frac{3}{2})$



$x>0$ since $t>0$ ✓

Note that

$$\left\{x=t+\frac{1}{t},\ y=t-\frac{1}{t},\ t>0\right\}$$

$$\{x=\sqrt{4+t^2},\ y=t,\ -\infty<t<\infty\}$$

$$\left\{x=2\sec t,\ y=2\tan t,\ -\frac{\pi}{2}<t<\frac{\pi}{2}\right\}$$

are all different parametrization for the same curve.