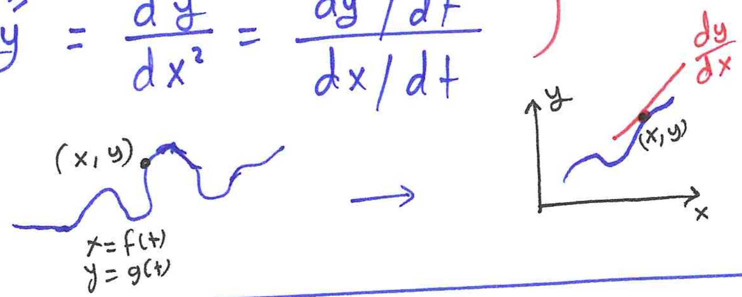
11.2 Calculus with Parametric Curves

* Parametric formulars for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$: • Given the parametric equations:

$$x = f(t), \quad y = g(t).$$

$$\left.\begin{aligned} y' &= \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \\ y'' &= \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} \end{aligned}\right\} *$$

• If x, y, y' are differentiable at $\underline{\underline{t}}$ with $\frac{dx}{dt} \neq 0$, then $*$ holds at any point $\underline{\underline{t}}$.



Exp • Find the tangent to the curve $x = t$, $y = \sqrt{t}$, $t = \frac{1}{4}$

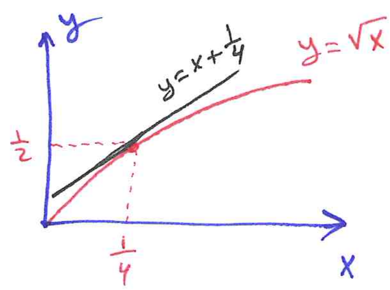
• The point is $\left(f(\tfrac{1}{4}), g(\tfrac{1}{4})\right) = \left(\tfrac{1}{4}, \tfrac{1}{2}\right)$

• The tangent line is $y - \frac{1}{2} = m(x - \frac{1}{4})$, where

the slope $m = \left.\frac{dy}{dx}\right|_{t=\frac{1}{4}} = \left.\frac{dy/dt}{dx/dt}\right|_{t=\frac{1}{4}} = \left.\frac{\frac{1}{2\sqrt{t}}}{1}\right|_{t=\frac{1}{4}} = 1$

$\Rightarrow$ The tangent line becomes $y - \frac{1}{2} = x - \frac{1}{4}$

$$y = x + \frac{1}{4}$$



• Find $\frac{d^2y}{dx^2}$ as a function of t

$$\frac{dy}{dx} = y' = \frac{dy/dt}{dx/dt} = \frac{1}{2\sqrt{t}}$$

$$\frac{d^2y}{dx^2} = y'' = \frac{dy'/dt}{dx/dt} = \frac{\frac{1}{2}(-\frac{1}{2})\, t^{-\frac{3}{2}}}{1}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{4\sqrt{t^3}}$$

• Find $\left.\frac{d^2y}{dx^2}\right|_{t=\frac{1}{4}} = \frac{-1}{4\sqrt{\frac{1}{64}}} = \frac{-1}{4\cdot\frac{1}{8}} = \frac{-1}{\frac{1}{2}} = -2$

Exp Find the slope of the curve "x and y are implicitly differentiable"

$x^3 + 2t^2 = 9 \quad , \quad 2y^3 - 3t^2 = 4$ at $t = 2$

• Note that when $\boxed{t=2}$ $\Rightarrow x^3 + 2(2)^2 = 9 \Rightarrow x^3 + 8 = 9$

$\Rightarrow x^3 = 1$

$\Rightarrow \boxed{x = 1}$

$\Rightarrow 2y^3 - 3(2)^2 = 4 \Rightarrow 2y^3 - 12 = 4$

$\Rightarrow 2y^3 = 16$

$\Rightarrow \boxed{y = 2}$

• The slope is:

$$m = \frac{dy/dt}{dx/dt}\Bigg|_{t=2}$$

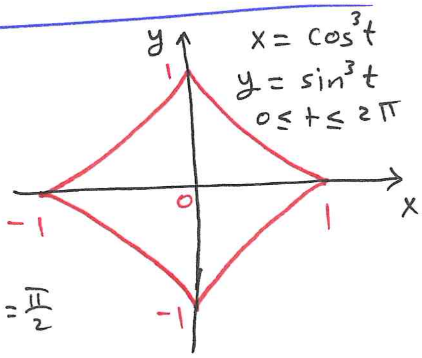
• $3x^2 \dfrac{dx}{dt} + 4t = 0 \Rightarrow \dfrac{dx}{dt} = \dfrac{-4t}{3x^2}$

• $6y^2 \dfrac{dy}{dt} - 6t = 0 \Rightarrow \dfrac{dy}{dt} = \dfrac{t}{y^2}$

$$= \frac{\frac{t}{y^2}}{\frac{-4t}{3x^2}}\Bigg|_{t=2}$$

$$= \frac{\frac{2}{4}}{\frac{-8}{3}} = -\frac{1}{2}\cdot\frac{3}{8} = \frac{-3}{16}$$

Exp Find the area enclosed by the astroid



$x = \cos^3 t$

$y = \sin^3 t$

$0 \le t \le 2\pi$

$$A = 4\int_0^1 y\,dx = 4\int_{\frac{\pi}{2}}^{0} \sin^3 t \left(3\cos^2 t(-\sin t)\right) dt$$

when $x = 0 \Rightarrow 0 = \cos^3 t \Rightarrow 0 = \cos t \Rightarrow t = \cos^{-1} 0 = \frac{\pi}{2}$

$x = 1 \Rightarrow 1 = \cos^3 t \Rightarrow 1 = \cos t \Rightarrow t = \cos^{-1} 1 = 0$

$$A = 12\int_0^{\frac{\pi}{2}} \sin^4 t \cos^2 t\,dt = 12\int_0^{\frac{\pi}{2}} \left(\frac{1-\cos 2t}{2}\right)^2 \left(\frac{1+\cos 2t}{2}\right) dt$$

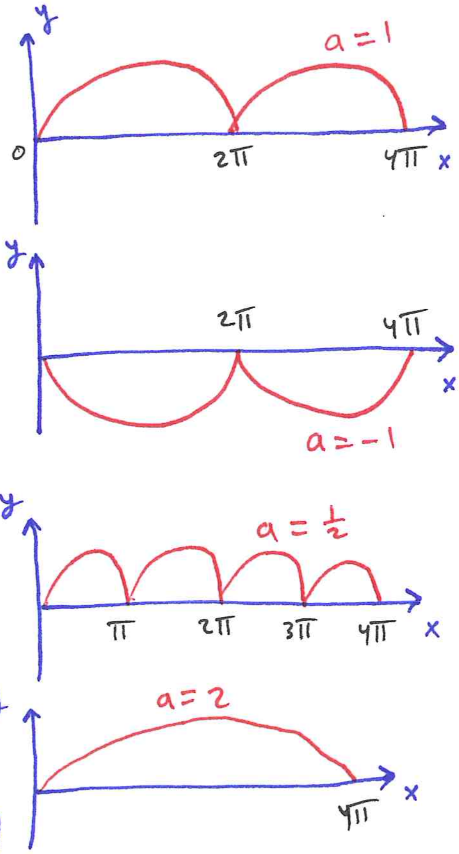
$$= \frac{12}{8}\int_0^{\frac{\pi}{2}} (1 - 2\cos 2t + \cos^2 2t)(1 + \cos 2t)\,dt$$

$$= \frac{3}{2}\int_0^{\frac{\pi}{2}} (1 - \cos 2t - \cos^2 2t + \cos^3 2t)\,dt = \cdots = \frac{3\pi}{8}.$$

Exp Find the area under one arch of the cycloid:

$$x = a(t - \sin t), \quad y = a(1 - \cos t)$$

when $a = 1$.



$$A = \int_0^{2\pi} y\,dx = \int_0^{2\pi} (1-\cos t)(1-\cos t)\,dt$$

$$= \int_0^{2\pi} (1-\cos t)^2\,dt = \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\,dt$$

$$= \int_0^{2\pi} \left(1 - 2\cos t + \frac{1+\cos 2t}{2}\right) dt$$

$$= 3\pi$$

Exp Find the area enclosed by the y-axis and the curve $x = t - t^2,\ y = 1 + e^{-t}, \quad 0 \le t \le 1$

$$A = \int_0^1 x\,dy = \int_0^1 (t-t^2)(-e^{-t})\,dt = \int_1^0 (t-t^2)\,e^{-t}\,dt$$

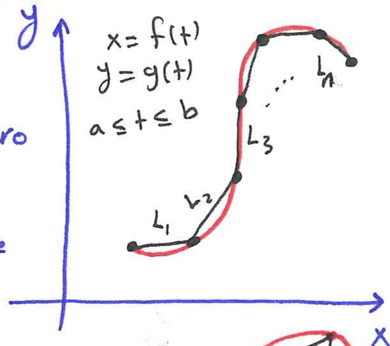
$(t-t^2)$	$(+)$	e^{-t}
$1-2t$	$(-)$	$-e^{-t}$
-2	$(+)$	e^{-t}
0		$-e^{-t}$

$$= \Big[(t^2 - t)\,e^{-t} - (1-2t)\,e^{-t} + 2\,e^{-t}\Big]_1^0 = (-1+2) - (e^{-1} + 2e^{-1})$$

$$= 1 - \frac{3}{e}$$

* length of a Curve defined parametrically:

If
- f' and g' are continuous and not simultaneously zero on $[a,b]$
- the curve traversed exactly once as t increase on $[a,b]$



then the length of the curve is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$\sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

$$L = \lim_{\|P\| \to 0} \sum_{k=1}^{n} L_k$$

Exp Find the length of the curve:

$$x = \frac{t^2}{2}, \quad y = \frac{(2t+1)^{\frac{3}{2}}}{3}, \quad 0 \le t \le 4$$

$$L = \int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^4 \sqrt{t^2 + 2t + 1}\, dt = \int_0^4 \sqrt{(t+1)^2}\, dt$$

$$= \int_0^4 |t+1|\, dt = \int_0^4 (t+1)\, dt = \left(\frac{t^2}{2} + t\right)\Big|_0^4 = 12$$

Area of Surfaces of Revolution

Given a smooth curve $x = f(t)$, $y = g(t)$, $a \le t \le b$ traversed exactly once as t increases from a to b.

The area of the surfaces generated by revolving the curve about coordinates axes are as follows:

* Revolution about x-axis $(y \geq 0)$: $S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

* Revolution about y-axis $(x \geq 0)$: $S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Exp Find the area of surface generating by revolving the curves

- $x = \cos t$, $y = 2 + \sin t$, $0 \le t \le 2\pi$, x-axis

$$S = \int_0^{2\pi} 2\pi (2 + \sin t)\sqrt{\sin^2 t + \cos^2 t}\, dt = 2\pi \int_0^{2\pi} (2 + \sin t)\, dt$$

$$= 2\pi \left[2t - \cos t\right]\Big|_0^{2\pi} = 2\pi\left[4\pi - 1 - (0 - 1)\right] = 8\pi^2$$