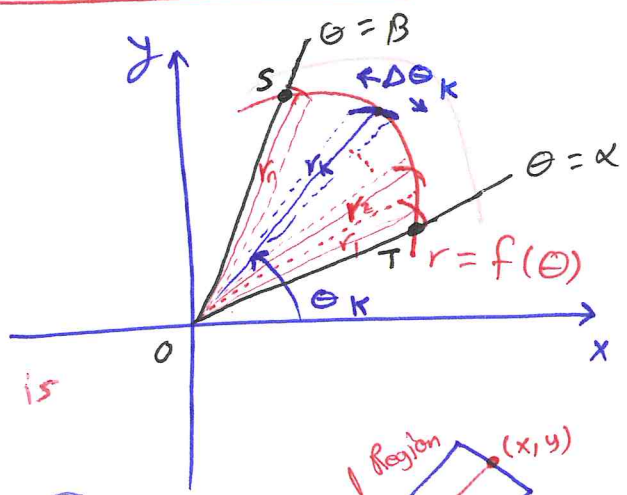


11.5 Areas and Lengths in Polar Coordinates

8

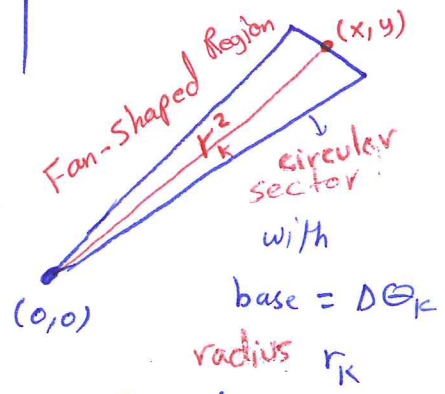
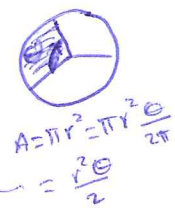
The region OTS is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$.



Area of the circular sector k is

$$A_k = \frac{1}{2} r_k^2 \Delta \theta_k$$

$$= \frac{1}{2} [f(\theta_k)]^2 \Delta \theta_k$$



The area of the region can be approximated by

$$\tilde{A} = \sum_{k=1}^n A_k = \sum_{k=1}^n \frac{1}{2} [f(\theta_k)]^2 \Delta \theta_k$$

The approximation is improved as $\|P\| \rightarrow 0$ "n to infinity"

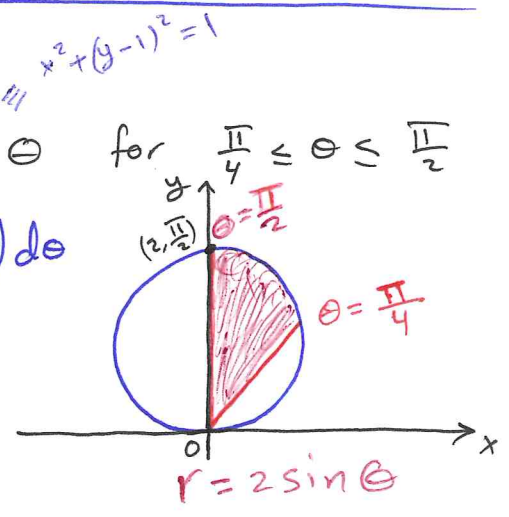
$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta = \lim_{\|P\| \rightarrow 0} \tilde{A}$$

Exp Find the areas of the region

(i) bounded by the circle $r = 2 \sin \theta$ for $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

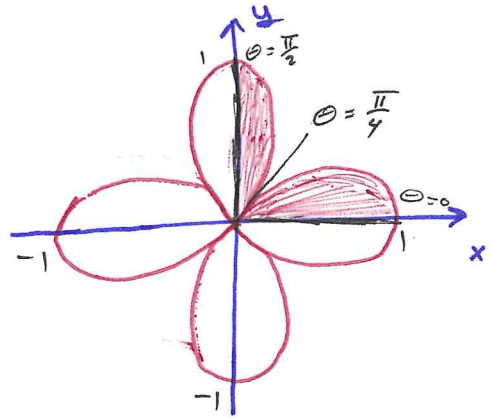
$$A = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2} [2 \sin \theta]^2 d\theta = 2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{1 - \cos 2\theta}{2} \right) d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 - \cos 2\theta) d\theta = \frac{\pi}{4} + \frac{1}{2}$$

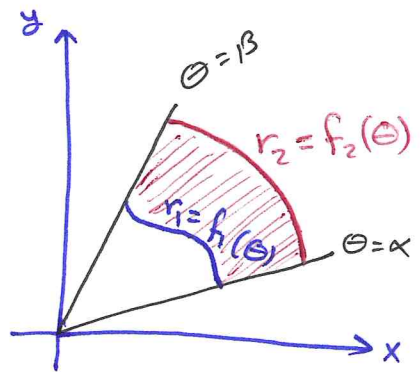


2] Inside one leaf of the four-leaved rose $r = \cos 2\theta$ (9)

$$\begin{aligned}
 A &= 2 \int_0^{\frac{\pi}{4}} \frac{1}{2} [\cos 2\theta]^2 d\theta \\
 &= \int_0^{\frac{\pi}{4}} \frac{1 + \cos 4\theta}{2} d\theta = \frac{\pi}{8} \\
 &= 2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2} [\cos 2\theta]^2 d\theta
 \end{aligned}$$



$$\begin{aligned}
 A &= \int_{\alpha}^{\beta} \frac{1}{2} r_2^2 d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2 d\theta \\
 &= \int_{\alpha}^{\beta} \frac{1}{2} (r_2^2 - r_1^2) d\theta
 \end{aligned}$$



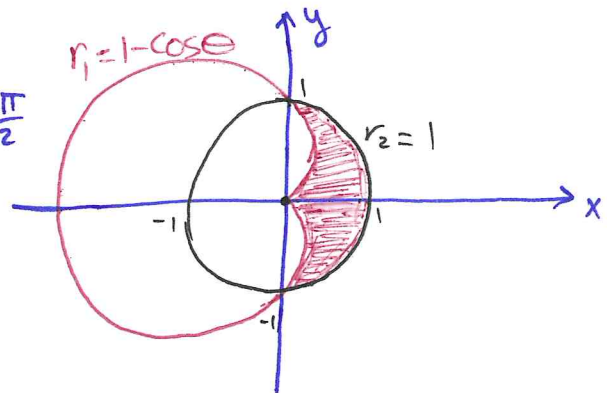
Exp Find the area that lies inside the circle $r=1$ and outside the cardioid $r=1-\cos\theta$

$$r=1=1-\cos\theta \Leftrightarrow \cos\theta=0 \Leftrightarrow \theta=\frac{\pi}{2}, -\frac{\pi}{2}$$

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} (r_2^2 - r_1^2) d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} [1^2 - (1-\cos\theta)^2] d\theta = 2 - \frac{\pi}{4}$$

$$= 2 \int_0^{\frac{\pi}{2}} \frac{1}{2} [1^2 - (1-\cos\theta)^2] d\theta = \int_0^{\frac{\pi}{2}} [1 - (1-\cos\theta)^2] d\theta = 2 - \frac{\pi}{4}$$



Length of a Polar Curve

(10)

Recall that $x = r \cos \theta = f(\theta) \cos \theta$
 $y = r \sin \theta = f(\theta) \sin \theta$ $\alpha \leq \theta \leq \beta$

are parametrization for the curve $r = f(\theta)$.

The length of r is $L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$

$$\frac{dx}{d\theta} = -r \sin \theta + \cos \theta \frac{dr}{d\theta}$$

$$\frac{dy}{d\theta} = r \cos \theta + \sin \theta \frac{dr}{d\theta}$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 \sin^2 \theta + \cos^2 \theta \left(\frac{dr}{d\theta}\right)^2 - 2r \sin \theta \cos \theta \frac{dr}{d\theta} + r^2 \cos^2 \theta + \sin^2 \theta \left(\frac{dr}{d\theta}\right)^2 + 2r \sin \theta \cos \theta \frac{dr}{d\theta}} d\theta$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Exp Find the length of the curve given by the spiral

$$r = \frac{e^{\theta}}{\sqrt{2}}, \quad 0 \leq \theta \leq \pi$$

$$L = \int_0^{\pi} \sqrt{\frac{e^{2\theta}}{2} + \frac{e^{2\theta}}{2}} d\theta = \int_0^{\pi} \sqrt{e^{2\theta}} d\theta = \int_0^{\pi} e^{\theta} d\theta$$
$$= e^{\pi} - 1$$