8.7 Improper Integrals (Type I and Type II)

Def: Improper Integrals of Type I are integrals with infinite limits of integration:

[1] $$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^{b} f(x)\,dx$$ where f is continous on $[a,\infty)$

[2] If f is continuous on $(-\infty, a)$, then

$$\int_{-\infty}^{a} f(x)\,dx = \lim_{b\to-\infty} \int_b^{a} f(x)\,dx$$

[3] If f is continuous on $(-\infty,\infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx \quad , c \in \mathbb{R}$$

In each case, if the limit is finite, then the improper integral converges and it is equal to this limit "Area"

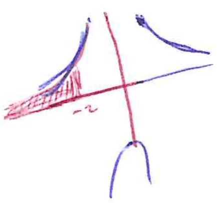
otherwise the improper integral diverges "infinit Area"

Exp ① $$\int_0^{\infty} \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_0^{b} \frac{dx}{x^2+1} = \lim_{b\to\infty} \tan^{-1}x \Big|_0^{b} = \lim_{b\to\infty} \tan^{-1} b = \frac{\pi}{2}$$

② $$\int_{-\infty}^{-2} \frac{2\,dx}{x^2-1} = \int_{-\infty}^{-2} \frac{2\,dx}{(x-1)(x+1)}$$

$$\frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$A = 1$
$B = -1$



$$= \int_{-\infty}^{-2} \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx$$

$$= \lim_{b\to-\infty} \int_b^{-2} \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx$$

$$= \lim_{b\to-\infty}\left(\ln|x-1| - \ln|x+1|\right)\Big|_b^{-2}$$

$$= \lim_{b\to-\infty}\left[\ln\left|\frac{x-1}{x+1}\right|\right]\Big|_b^{-2} = \lim_{b\to-\infty}\left[\ln\left|\frac{-3}{-1}\right| - \ln\left|\frac{b-1}{b+1}\right|\right]$$

$$= \ln 3 - \ln \lim_{b\to-\infty}\frac{b-1}{b+1} = \ln 3 - \ln 1 = \ln 3$$

Note that $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \int_{-\infty}^{0}\frac{dx}{1+x^2} + \int_{0}^{\infty}\frac{dx}{1+x^2} = \frac{\pi}{2}+\frac{\pi}{2} = \pi$

Exp $\displaystyle\int_0^{\infty}\frac{16\tan^{-1}x}{1+x^2}\,dx = \lim_{b\to\infty}\int_0^{b}\frac{16\tan^{-1}x}{1+x^2}\,dx$

$u = \tan^{-1}x$

$du = \dfrac{1}{1+x^2}\,dx$

$$= \lim_{b\to\infty}\int_0^{\tan^{-1}b}16\,u\,du = \lim_{b\to\infty} 8u^2\Big|_0^{\tan^{-1}b}$$

$$= \lim_{b\to\infty} 8\left[(\tan^{-1}b)^2 - (\tan^{-1}0)^2\right] = 8\left(\frac{\pi}{2}\right)^2 = 2\pi^2$$

Def Improper Integrals of Type II are integrals of functions that become infinite at a point within the interval of integration (Vertical Asymptotes)

$\boxed{1}$ If f is discontinuous at a, then $\displaystyle\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx$

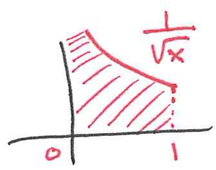
$\boxed{2}$ If f is discontinuous at b, then $\displaystyle\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx$

$\boxed{3}$ If f is discontinuous at c, where $a<c<b$, then $\displaystyle\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

In each case, if the limit is finite, then the improper integral converges and it is equal to this limit "Area".

Otherwise the improper integral diverges "Infinite area"

Exp $$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{c\to 0^+} \int_c^1 \frac{dx}{\sqrt{x}} = \lim_{c\to 0^+} 2\sqrt{x}\Big|_c^1$$



$$= \lim_{c\to 0^+} \left(2 - 2\sqrt{c}\right) = 2 - 0 = 2$$

Exp $$\int_0^4 \frac{dx}{\sqrt{4-x}} = \lim_{b\to 4^-} \int_0^b \frac{dx}{\sqrt{4-x}} = \lim_{b\to 4^-} -2\sqrt{4-x}\Big|_0^b$$

$$= \lim_{b\to 4^-} \left[-2\sqrt{4-b} + 4\right] = 4$$

Exp* For what values of p does the integral $\int_1^\infty \frac{dx}{x^p}$ converge?

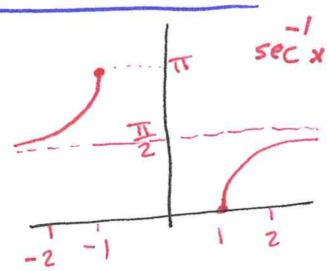
$$p \neq 1 \Rightarrow \int_1^\infty \frac{dx}{x^p} = \lim_{c\to\infty} \int_1^c x^{-p}\,dx = \lim_{c\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^c$$

$$= \lim_{c\to\infty} \left[\frac{c^{1-p}}{1-p} - \frac{1}{1-p}\right] = \begin{cases} \frac{1}{p-1}, & p>1 \\ \infty, & p<1 \end{cases}$$

diverges

$$\boxed{p=1} \Rightarrow \int_1^\infty \frac{dx}{x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x} = \lim_{b\to\infty} \ln|x|\Big|_1^b = \lim_{b\to\infty} \ln|b| = \infty$$

Exp $$\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \int_1^2 \frac{dx}{x\sqrt{x^2-1}} + \int_2^\infty \frac{dx}{x\sqrt{x^2-1}}$$



$$= \lim_{b\to 1^+} \int_b^2 \frac{dx}{x\sqrt{x^2-1}} + \lim_{c\to\infty} \int_2^c \frac{dx}{x\sqrt{x^2-1}}$$

$$= \lim_{b\to 1^+} \left[\sec^{-1}|x|\right]_b^2 + \lim_{c\to\infty} \left[\sec^{-1}|x|\right]_2^c$$

$$= \lim_{b\to 1^+} \left[\cancel{\underbrace{\sec^{-1} 2}_{\frac{\pi}{3}}} - \sec^{-1} b\right] + \lim_{c\to\infty} \left[\sec^{-1} c - \cancel{\underbrace{\sec^{-1} 2}_{\frac{\pi}{3}}}\right]$$

$$= \lim_{b\to 1^+} -\sec^{-1} b + \lim_{c\to\infty} \sec^{-1} c$$

$$= 0 + \frac{\pi}{2} = \frac{\pi}{2}$$

Tests for Convergence and Divergence

Th (Direct Comparasion Test)

Let f and g be continuous on $[a, \infty)$ with

$$0 \le f(x) \le g(x) \text{ for all } x \ge a.$$

* If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.

* If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.

Exp $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converges because $0 \le \underbrace{\frac{\sin^2 x}{x^2}}_{f(x)} \le \underbrace{\frac{1}{x^2}}_{g(x)}$ on $[1, \infty)$

and $\int_1^\infty \frac{1}{x^2}$ converges "see exp *"

Thus, by the Direct Comparasion Test $\int_1^\infty \frac{\sin^2}{x^2}\,dx$ converges

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Exp $\int_1^\infty \frac{1}{\sqrt{x^2 - 0.1}}\,dx$ diverges because $\underbrace{\frac{1}{\sqrt{x^2 - 0.1}}}_{g(x)} \ge \underbrace{\frac{1}{x}}_{f(x)}$ on $[1, \infty)$

and $\int_1^\infty \frac{1}{x}$ diverges "see exp *"

Thus, by the Direct Comparision Test, $\int_1^\infty \frac{dx}{\sqrt{x^2 - 0.1}}$ diverges

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Exp $\int_0^\pi \frac{dt}{\sqrt{t} + \sin t}$ converges because $0 \le \underbrace{\frac{1}{\sqrt{t} + \sin t}}_{f(x)} \le \underbrace{\frac{1}{\sqrt{t}}}_{g(t)}$ on $[0, \pi]$

and $\int_0^\pi \frac{dt}{\sqrt{t}} = \lim_{b \to 0^+} \int_b^\pi t^{-\frac{1}{2}}\,dt = \lim_{b \to 0^+} 2\sqrt{t}\,\Big|_b^\pi = \lim_{b \to 0^+} 2\sqrt{\pi} - 2\sqrt{b}$

$= 2\sqrt{\pi}$

which converges, Thus, by the Direct Comparasion test, $\int_0^\pi \frac{dt}{\sqrt{t} + \sin t}$ converges.

2 Th (limit Comparision Test)

If • f and g are positive and continous functions on $[a, \infty)$

• $\lim_{x \to \infty} \frac{f(x)}{g(x)} = L$ where $0 < L < \infty$,

then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ both converge or both diverge.

Exp $\int_1^\infty \frac{dx}{1+x^2}$

• let $f(x) = \frac{1}{x^2}$, $g(x) = \frac{1}{1+x^2}$

• f and g are positive continuous on $[1, \infty)$

• $\lim_{x \to \infty} \frac{f}{g} = \lim_{x \to \infty} \frac{\frac{1}{x^2}}{\frac{1}{1+x^2}} = \lim_{x \to \infty} \frac{1+x^2}{x^2} = 1$ "finite +"

and $\int_1^\infty \frac{dx}{x^2}$ $= \frac{1}{2-1} = 1$ converges "see exp*". Thus, by the Limit Comparasion Test

$\Rightarrow \int_1^\infty \frac{dx}{1+x^2}$ converges. $\frac{\pi}{4}$

Exp $\int_2^\infty \frac{dx}{\sqrt{x-1}}$

• Let $f(x) = \frac{1}{\sqrt{x-1}}$, $g(x) = \frac{1}{\sqrt{x}}$

• f and g are positive continuous on $[2, \infty)$

• $\lim_{x \to \infty} \frac{f}{g} = \lim_{x \to \infty} \frac{\frac{1}{\sqrt{x-1}}}{\frac{1}{\sqrt{x}}} = \lim_{x \to \infty} \frac{\sqrt{x}}{\sqrt{x-1}} = \lim_{x \to \infty} \frac{1}{\sqrt{1-\frac{1}{x}}} = 1$

and $\int_2^\infty \frac{dx}{\sqrt{x}} = \lim_{b \to \infty} \int_2^b x^{-\frac{1}{2}}\,dx = \lim_{b \to \infty} 2\sqrt{x}\Big|_2^b = \lim_{b \to \infty} \left[2\sqrt{b} - 2\sqrt{2}\right] = +\infty$

which diverges $\Rightarrow \int_2^\infty \frac{dx}{\sqrt{x-1}}$ diverges by the limit Comparasion Test.

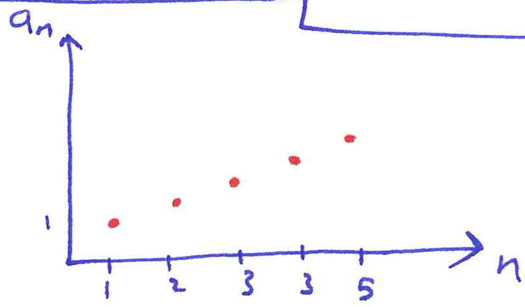
$\boxed{10.1}$ Sequences

* A sequence is a list of numbers $a_1, a_2, \dots, a_n, \dots$

- where a_i are numbers with index i "order"
- it can be finite or infinite
- it is a function that sends 1 to a_1
  2 to a_2
  $\vdots$
  n to a_n "the n^{th} term"

Exp $a_n = \sqrt{n}$

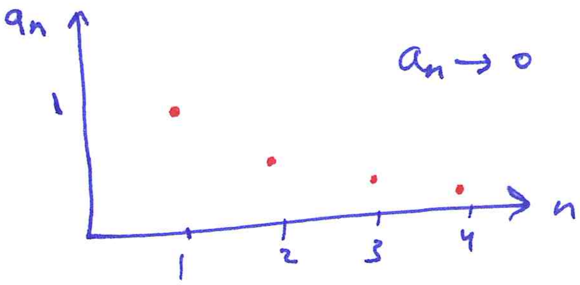
$a_1 = 1$
$a_2 = \sqrt{2}$
$a_3 = \sqrt{3}$
$\vdots$
$a_n = \sqrt{n}$
$\vdots$



$a_n \to \infty$ as $n \to \infty$

Exp $a_n = \frac{1}{n}$

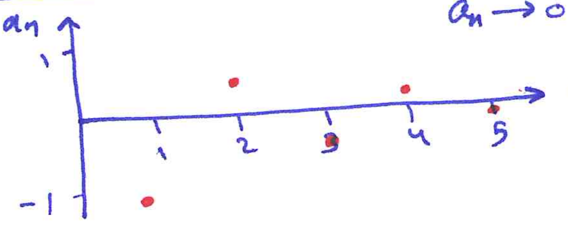
$a_1 = 1$
$a_2 = \frac{1}{2}$
$a_3 = \frac{1}{3}$
$\vdots$



$a_n \to 0$ as $n \to \infty$

Exp $a_n = (-1)^n \frac{1}{n}$

$a_1 = -1$
$a_2 = +\frac{1}{2}$
$a_3 = -\frac{1}{3}$
$\vdots$



$a_n \to 0$ as $n \to \infty$

By Sandwich Th.

$$\underbrace{-\frac{1}{n}}_{\to 0} \le (-1)^n \frac{1}{n} \le \underbrace{\frac{1}{n}}_{\to 0}$$

Def: The sequence $\{a_n\}$ converges to the number L "$a_n \to L$" $\equiv$ "$\lim_{n\to\infty} a_n = L$"

if for every number $\epsilon > 0$, there exists an integer N s.t

for all $n > N \Rightarrow |a_n - L| < \epsilon$.

If such number L does not exist, we say the sequence $\{a_n\}$ diverges. $\to \infty$, $\to -\infty$

$\lim_{n\to\infty} a_n = \infty$, $\lim a_n = -\infty$

Exp ① Find $\lim_{n\to\infty} \frac{1}{n} = 0$

② Show that $\lim_{n\to\infty} \frac{1}{n} = 0$ Let $\epsilon > 0$, we need to show that there exist an integer N such that for all

$$n > N \Rightarrow |a_n - L| < \epsilon$$
$$|\tfrac{1}{n} - 0| < \epsilon \Leftrightarrow |\tfrac{1}{n}| < \epsilon$$

This implication will hold if $\frac{1}{n} < \epsilon \Leftrightarrow n > \frac{1}{\epsilon}$

Take N to be any integer greater than $\frac{1}{\epsilon}$

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③ Find $\lim_{n\to\infty} K = K$

④ Show that $\lim_{n\to\infty} K = K$ Let $\epsilon > 0$, we need to show that $\exists$ an integer N s.t for all

$$n > N \Rightarrow |a_n - L| < \epsilon$$
$$|K - K| < \epsilon \Leftrightarrow 0 < \epsilon$$

N can be any positive integer.

Th Assume that $\{a_n\}$ and $\{b_n\}$ are sequences of real #'s, and let A and B be real #'s.

If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then

① $\lim_{n\to\infty} (a_n + b_n) = A + B$ ..... Sum Rule

② $\lim_{n\to\infty} (a_n - b_n) = A - B$ .... Difference Rule

③ $\lim_{n\to\infty} (K b_n) = K B$, ($K$ is any number)... Constant Multiple Rule

④ $\lim_{n\to\infty} (a_n b_n) = A B$ ..... Product Rule

⑤ $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ " $B \neq 0$ " .... Quotient Rule

Exp ① $\lim_{n\to\infty}\left(\frac{-\sqrt{3}}{n}\right) = -\sqrt{3}\lim_{n\to\infty}\frac{1}{n} = -\sqrt{3}\cdot 0 = 0$

② $\lim_{n\to\infty}\left(\frac{2n+5}{3n}\right) = \lim_{n\to\infty}\left(\frac{2}{3}+\frac{5}{n}\right) = \frac{2}{3}$

③ $\lim_{n\to\infty}\frac{n-2n^3}{n^3} = \lim_{n\to\infty}\left(\frac{1}{n^2}-2\right) = -2 + \lim_{n\to\infty}\frac{1}{n}\lim_{n\to\infty}\frac{1}{n}$

$= -2 + 0\cdot 0 = -2$

④ $\lim_{n\to\infty}\frac{3+\sqrt{8}\,n^5}{3+\sqrt{2}\,n^5} = \lim_{n\to\infty}\frac{\frac{3}{n^5}+\sqrt{8}}{\frac{3}{n^5}+\sqrt{2}} = \frac{0+\sqrt{8}}{0+\sqrt{2}} = 2$

Th (Sandwich Th.)

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers with $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$. If $a_n \le b_n \le c_n$ for all n beyond some number N, then $\lim_{n\to\infty} b_n = L$.

Exp ① $\lim_{n\to\infty}\frac{\sin n}{n} \to 0$ because $-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$

② $(-1)^n\frac{1}{n} \to 0$ because $-\frac{1}{n} \le (-1)^n\frac{1}{n} \le \frac{1}{n}$

Th 5 $\boxed{1}$ $\lim_{n\to\infty}\frac{\ln n}{n} = 0$ $\qquad \lim_{n\to\infty}\frac{\frac{1}{n}}{1} = \lim_{n\to\infty}\frac{1}{n} = 0$

$\boxed{2}$ $\lim_{n\to\infty}\sqrt[n]{n} = 1$ $\qquad \lim_{n\to\infty} e^{\ln n^{\frac{1}{n}}} = \lim_{n\to\infty} e^{\frac{\ln n}{n}} = e^0 = 1$

$\boxed{3}$ $\lim_{n\to\infty} x^{\frac{1}{n}} = 1 \quad (x>0)$ $\qquad \lim_{n\to\infty} e^{\ln x^{\frac{1}{n}}} = \lim_{n\to\infty} e^{\frac{\ln x}{n}} = e^0 = 1$

$\left(\frac{1}{2}\right)^n \to 0$ as $n\to\infty$ / $\left(-\frac{1}{2}\right)^n = (-1)^n\frac{1}{2^n} \to 0$ by Sandwich

$\boxed{4}$ $\lim_{n\to\infty} x^n = 0 \quad (|x|<1)$

$\to x$ is fixed as $n\to\infty$

$\boxed{5}$ $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ $\qquad = \lim_{n\to\infty} e^{\ln\left(1+\frac{x}{n}\right)^n} = \lim_{n\to\infty} e^{\frac{\ln\left(1+\frac{x}{n}\right)}{\frac{1}{n}}} = \lim_{n\to\infty} e^{\frac{\frac{-\frac{x}{n^2}}{1+\frac{x}{n}}}{\frac{-1}{n^2}}}$

$= e^x$

$\boxed{6}$ $\lim_{n\to\infty}\frac{x^n}{n!} = 0$, (any x) $\qquad \frac{\infty}{\infty} \Rightarrow \frac{n x^{n-1}}{(n!)'}, \frac{n(n-1)x^{n-2}}{(n!)''}, \dots, \frac{\text{constant}}{(n!)^{(n)}} = 0$

or • $0 \le \frac{x x \cdots x}{n(n-1)\cdots 1} \le \frac{x}{n} \to 0$ or $\frac{x}{n} \le \frac{x x \cdots x}{n(n-1)\cdots 1} \le 0$ ($\frac{x}{n}\to 0$) Sandwich

Exp ① $\lim\limits_{n\to\infty} \frac{\ln n^3}{3n} = \lim\limits_{n\to\infty} \frac{\ln n}{n} = 0$ $\boxed{1}$

② $\lim\limits_{n\to\infty} \sqrt[n]{n^3} = \lim\limits_{n\to\infty} n^{\frac{3}{n}} = \lim\limits_{n\to\infty} \left(n^{\frac{1}{n}}\right)^3 = 1^3 = 1$ $\boxed{2}$

③ $\lim\limits_{n\to\infty} \sqrt[n]{\pi n} = \lim\limits_{n\to\infty} \pi^{\frac{1}{n}} \lim\limits_{n\to\infty} n^{\frac{1}{n}} = (1)(1) = 1$
$\boxed{3}$ $\boxed{2}$

④ $\lim\limits_{n\to\infty} \frac{\pi^{-n}}{e^{-n}} = \lim\limits_{n\to\infty} \left(\frac{\pi}{e}\right)^{-n} = \lim \left(\frac{e}{\pi}\right)^n = 0$ $\boxed{4}$

⑤ $\lim\limits_{n\to\infty} \left(\frac{n+1}{n-1}\right)^n = \lim\limits_{n\to\infty} \left(\frac{n-1+2}{n-1}\right)^n = \lim\limits_{n\to\infty} \left(1 + \frac{2}{n-1}\right)^n$

$u = n-1$
$1+u = n$

$= \lim\limits_{u\to\infty} \left(1 + \frac{2}{u}\right)^{u+1}$

$= \lim\limits_{u\to\infty} \left(1 + \frac{2}{u}\right) \lim\limits_{u\to\infty} \left(1 + \frac{2}{u}\right)^u$ $\boxed{5}$

$= (1)(e^2) = e^2$ (see the book) for another way.

Exp Find a formula for the n^{th} term of the sequence

① $1, -4, 9, -16, 25, \ldots$ $\quad a_n = (-1)^{n+1} \cdot n^2, \quad n = 1, 2, 3, \ldots$
$n \geq 1$

② $0, 3, 8, 15, 24, \ldots$ $\quad a_n = n^2 - 1, \quad n \geq 1$

Exp (Recursive Defined Sequence) Assume the following sequence converges, find it's limit.

$a_1 = 1, \quad a_{n+1} = \frac{1}{2} a_n$

$a_2 = \frac{1}{2} a_1 = \frac{1}{2}$

$a_3 = \frac{1}{2} a_2 = \frac{1}{2}\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2$

$a_4 = \frac{1}{2} a_3 = \frac{1}{2}\left(\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^3$

$a_5 = \frac{1}{2} a_4 = \frac{1}{2}\left(\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^4$

$\vdots$

$a_n = \left(\frac{1}{2}\right)^{n-1}$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(\frac{1}{2}\right)^{n-1}$

$= 2 \lim\limits_{n\to\infty} \left(\frac{1}{2}\right)^n$

$= 2 \cdot 0$

$= 0$

10.2 Infinite Series

* An infinite series is the sum of an infinite sequence of numbers $a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n$

where
- a_n is the n^{th} term of the series
- $s_1 = a_1$ is the 1^{st} partial sum of the series
- $s_2 = a_1 + a_2$ is the 2^{nd} partial sum of the series

  $\vdots$
- $s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$ is the n^{th} partial sum of series.
- If the sequence of partial sums converges to a limit L then we say the series converges and we write $a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L$
- If the sequence of partial sums of the serie does not converge, then we say the series diverges.

Exp $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$

<u>Partial sums</u>

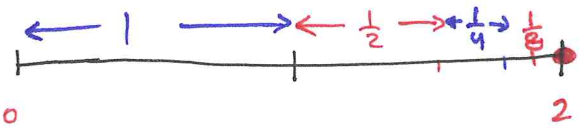
First $s_1 = 1 = 2 - 1$

second $s_2 = 1 + \frac{1}{2} = 2 - \frac{1}{2}$

Third $s_3 = 1 + \frac{1}{2} + \frac{1}{4} = 2 - \frac{1}{4}$

$\vdots$

nth $s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} = 2 - \frac{1}{2^{n-1}}$



- Note that this sequence of partial sums converges to 2 because $\lim_{n\to\infty} s_n = 2 - 0 = 2$.
- Thus, we say the sum of the infinite series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 2$

Geometric Series

Geometric series are series of the form:

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} a r^{n-1} \quad *$$

where • a and r are fixed real numbers

• $a \neq 0$

• r is called the ratio and can be positive:

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots \qquad r = \frac{1}{2}, \quad a = 1$$

or negative:

$$1 - \frac{1}{3} + \frac{1}{9} - \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots \qquad r = \frac{-1}{3}, \quad a = 1$$

To determine the convergence and divergence of the geometric series $*$; we consider 3 cases:

① • If $r=1$, then the nth partial sum of the geometric series is $S_n = a + a(1) + a(1)^2 + \cdots + a(1)^{n-1} = na$ and the series diverges because $\lim_{n\to\infty} S_n = \pm\infty$ depending on the sign of a.

② • If $r=-1$, then the nth partial sum of the series is $S_n = \cancel{a} - \cancel{a} + \cancel{a} - \cancel{a} + a + \cdots a(-1)^{n-1} = \begin{cases} 0 & \text{if } n \text{ even} \\ a & \text{if } n \text{ odd} \end{cases}$

Thus, the series diverges because the nth partial sum alternate between a and 0.

③ • If $r \neq 1$ and $r \neq -1$ (i.e $|r| \neq 1$) then we can determine the convergence or divergence as follows:

$$S_n = a + \cancel{ar} + \cancel{ar^2} + \cancel{\cdots} + \cancel{ar^{n-2}} + ar^{n-1}$$

$$rS_n = \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cdots + \cancel{ar^{n-1}} + ar^n$$

$$S_n - rS_n = a - ar^n \Leftrightarrow S_n(1-r) = a(1-r^n) \Leftrightarrow$$

$$S_n = \frac{a(1-r^n)}{1-r}, \quad r \neq 1$$

- If $|r|<1$, then $r^n \to 0$ as $n \to \infty$

$\Rightarrow \lim_{n\to\infty} S_n = \frac{a}{1-r}$. Thus $\sum_{n=1}^{\infty} ar^{n-1}$ converges to $\frac{a}{1-r}$

- If $|r|>1$, then $r^n \to \infty$ and the series diverges

---

* If $|r|<1$, the geometric series

$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $\frac{a}{1-r}$: $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$, $\underline{|r|<1}$

* If $|r| \geq 1$, the series diverges

Exp: $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = \frac{1}{1-\frac{1}{2}} = 2$

Exp: $1 - \frac{1}{3} + \frac{1}{9} - \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^{n-1} = \frac{1}{1-\frac{-1}{3}} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$

Exp: Express the repeating decimal numbers as the ratio of two integers:

① $0.\overline{23} = 0.23\ 23\ 23 \cdots = \frac{23}{100} + \frac{23}{(100)^2} + \frac{23}{(100)^3} + \cdots = \sum_{n=1}^{\infty} \frac{23}{100}\left(\frac{1}{100}\right)^{n-1}$

$= \frac{\frac{23}{100}}{1-\frac{1}{100}} = \frac{\frac{23}{100}}{\frac{99}{100}} = \frac{23}{99}$

② $0.\overline{7} = 0.777\cdots = \frac{7}{10} + \frac{7}{(10)^2} + \cdots$

$= \frac{\frac{7}{10}}{1-\frac{1}{10}} = \frac{\frac{7}{10}}{\frac{9}{10}} = \frac{7}{9}$

③ $0.0\overline{6} = 0.0666\cdots = \frac{6}{100} + \frac{6}{1000} + \frac{6}{10000} + \cdots$

$= \frac{\frac{6}{100}}{1-\frac{1}{10}} = \frac{\frac{6}{100}}{\frac{9}{10}} = \frac{6}{90} = \frac{1}{15}$

The nth Term Test for Divergent Series

* If $\lim_{n\to\infty} a_n$ fails to exist or is different from zero, then $\sum_{n=1}^{\infty} a_n$ diverges

Exp ① The series $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$

② The series $\sum_{n=1}^{\infty} \sqrt{n}$ diverges because $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{n} = \infty$

③ The series $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\to\infty} (-1)^{n+1}$ does not exist

④ The series $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges because $\lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0$

Th 7 If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Note that Th7 does not say that if $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ converges : exp $\sum_{n=1}^{\infty} \frac{1}{n}$ "harmonic" series

Th 8 If $\sum_{n=1}^{\infty} a_n = A$ and $\sum b_n = B$ are convergent series; then

1. Sum Rule : $\sum (a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. Difference Rule $\sum (a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. Constant Multiple Rule: $\sum k\, a_n = k \sum a_n = kA$, $k \in \mathbb{R}$

Note that ① Every nonzero constant multiple of divergent series is divergent

② If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum (a_n + b_n)$ and $\sum (a_n - b_n)$ both diverges.

"Telescoping Series"

Exp Find a formula for the n^{th} partial sum of the following series and use it to determine if the series converges or diverges. If the series converges find the sum.

① $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$

$$S_n=\left(1-\cancel{\frac{1}{\sqrt{2}}}\right)+\left(\cancel{\frac{1}{\sqrt{2}}}-\cancel{\frac{1}{\sqrt{3}}}\right)+\left(\cancel{\frac{1}{\sqrt{3}}}-\cancel{\frac{1}{\sqrt{4}}}\right)+\cdots+\left(\cancel{\frac{1}{\sqrt{n-1}}}-\cancel{\frac{1}{\sqrt{n}}}\right)+\left(\cancel{\frac{1}{\sqrt{n}}}-\frac{1}{\sqrt{n+1}}\right)$$

$S_n=1-\frac{1}{\sqrt{n+1}}$ $\Rightarrow \lim_{n\to\infty} S_n=1$. Thus, the series converges to 1 i.e $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)=1$

② $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ "we use partial fraction"

$$\frac{1}{n(n+1)}=\frac{A}{n}+\frac{B}{n+1}, \quad A=1, \; B=-1$$

$$\sum_{n=1}^{n}\left[\frac{1}{n}-\frac{1}{n+1}\right]=\left(1-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\cdots+\left(\cancel{\frac{1}{n-1}}-\cancel{\frac{1}{n}}\right)+\left(\cancel{\frac{1}{n}}-\frac{1}{n+1}\right)$$

$$S_n=1-\frac{1}{n+1}$$

$\lim_{n\to\infty} S_n=1$. Thus, the series converges to 1.

i.e $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1$

Exp Find the sum of the following series

$$\sum_{n=0}^{\infty}\left(\frac{5}{2^n}+\frac{1}{3^n}\right)=(5+1)+\left(\frac{5}{2}+\frac{1}{3}\right)+\left(\frac{5}{4}+\frac{1}{9}\right)+\left(\frac{5}{8}+\frac{1}{27}\right)+\cdots$$

$$=\left[5+\frac{5}{2}+\frac{5}{4}+\frac{5}{8}+\cdots\right]+\left[1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots\right]$$

$$=\frac{5}{1-\frac{1}{2}}+\frac{1}{1-\frac{1}{3}}$$

$$=\frac{5}{\frac{1}{2}}+\frac{1}{\frac{2}{3}}$$

$$=10+\frac{3}{2}=\frac{23}{2}$$

10.3 The Integral Test

Corollary: A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges iff its partial sums (s_n) are bounded from above.

Exp $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^n + \cdots$ geometric series with $r = \frac{1}{2} < 1$

$$= \frac{1/2}{1 - \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

Note that $s_n \leq 1 \quad \forall n = 1, 2, 3, \ldots$

That is
$$s_1 = \frac{1}{2}$$
$$s_2 = \frac{1}{2} + \left(\frac{1}{2}\right)^2$$
$$s_3 = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3$$
$$\vdots$$
$$s_n = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^n.$$

note that since the serie converge $\Rightarrow a_n \to 0$ as $n \to \infty$

Exp $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ The harmonic series is divergent.

$$= 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> \frac{2}{4} = \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{> \frac{4}{8} = \frac{1}{2}} + \underbrace{\left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right)}_{> \frac{8}{16} = \frac{1}{2}} + \cdots$$

The sequence of the partial sums is not bounded above because we don't have $s_n \leq s_{n+1}$.

- Thus, the harmonic series diverges to ∞. The process is very slow. That is after 178 million terms, its partial sum is 20.

Thm "The Integral Test"

Consider the series $\sum_{n=K}^{\infty} a_n$, where

- a_n is a sequence of postive terms
- $a_n = f(n)$ is s.t f is continuous, positive, decreasing on $[K, \infty)$.

Then the series $\sum_{n=K}^{\infty} a_n$ and the integral $\int_K^{\infty} f(x)\,dx$ both converges or both diverges.

Exp Does the following series converge / diverge?

① $\sum_{n=1}^{\infty} \frac{1}{n^2}$ $(a_n \to 0$ as $n \to \infty)$ so it may converge

$f(x) = \frac{1}{x^2}$ is continous, positive, decreasing function on $[1, \infty)$

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2} = \frac{1}{2-1} = 1$$

"by exp.*" note that this is not sum we don't know the sum.

Thus, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the integral test.

② The p- series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$ } "by exp*"

③ $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ $(a_n \to 0$ as $n \to \infty)$ so it may converge

$f(x) = \frac{1}{x^2+1}$ is continous, positive, decreasing function on $[1, \infty)$

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2+1} = \lim_{b\to\infty} \tan^{-1}x \Big|_1^b$$

$$= \lim_{b\to\infty} \left[\tan^{-1} b - \tan^{-1} 1\right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Thus, the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges by the integral test.

④ $\sum_{n=1}^{\infty} n \sin(\frac{1}{n})$ diverges by the n^{th} term test

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = \lim_{x\to 0} \frac{\sin x}{x} = 1 \neq 0$$

⑤ $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ $(a_n \to 0$ as $n \to \infty)$ so it may converge

$f(x) = \frac{1}{2x-1}$ is continous, positive, decreasing function on $[1, \infty)$

$$\int_1^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_1^b \frac{dx}{2x-1} = \lim_{b\to\infty} \frac{1}{2} \ln|2x-1| \Big|_1^b = \lim_{b\to\infty} \frac{1}{2} \ln(2b-1) = \infty$$

Thus, the series diverges by the Integral test.

$\boxed{10.4}$ (Direct) Comparasion Test

Th 10 "Comparasion Test"

- Let $\sum a_n$, $\sum c_n$, $\sum d_n$ be series with nonnegative terms.
- Suppose that $d_n \le a_n \le c_n$ for all $n > N$.

(a) If $\sum c_n$ converges, then $\sum a_n$ also converges.

(b) If $\sum d_n$ diverges, then $\sum a_n$ also diverges.

Exp Use the Comparasion Test to determine if the following series converges or diverges?

① $\displaystyle\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2} \le 2\sum_{n=1}^{\infty} \frac{1}{n^2}$ → This is p-series with $p=2$ which converges

Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2}$ converges by the (Direct) Comparsion Test

② $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{3n-1}$ diverges because $\displaystyle\lim_{n\to\infty} a_n \ne 0$

$$\lim_{n\to\infty} \frac{2n}{3n-1} = \frac{2}{3} \ne 0$$

③ $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}} \ge \sum_{n=1}^{\infty} \frac{1}{n}$ → divergent series "harmonic series"

$3n > n+\sqrt{n} \Leftrightarrow$

$n+n+n > n+\sqrt{n}+0 \Leftrightarrow n > \frac{n+\sqrt{n}}{3}$

$\frac{1}{n} < \frac{3}{n+\sqrt{n}}$

Thus $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$ diverges by the comparsion Test.

④ $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n < \sum_{n=1}^{\infty} \left(\frac{n}{3n}\right)^n = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ → geometric series which converges

Thus, $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$ converges by the Comparsion Test

The limit Comparsion Test

Th (Limit Comparsion Test)

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$

1. If $\lim\limits_{n \to \infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converges or both diverge.

2. If $\lim\limits_{n \to \infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

3. If $\lim\limits_{n \to \infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Exp Use the limit Comparasion Test to determine whether the following series converges or diverges:

① $\sum\limits_{n=1}^{\infty} \frac{n-2}{n^3 - n^2 + 3}$ $\qquad a_n = \frac{n-2}{n^3 - n^2 + 3} \qquad b_n = \frac{1}{n^2}$

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\frac{n-2}{n^3 - n^2 + 3}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{n^3 - 2n^2}{n^3 - n^2 + 3} = 1 > 0$$

and $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges "p-series with $p=2$".

Thus, $\sum\limits_{n=1}^{\infty} \frac{n-2}{n^3 - n^2 + 3}$ converges by the limit Comparsion Test.

② $\sum\limits_{n=1}^{\infty} \frac{1}{1 + \ln n}$ $\qquad a_n = \frac{1}{1 + \ln n} \qquad b_n = \frac{1}{n}$

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\frac{1}{1+\ln n}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n}{1 + \ln n} = \lim_{n \to \infty} \frac{1}{\frac{1}{n}} = \infty$$

and $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges "harmonic series"

Thus, $\sum\limits_{n=1}^{\infty} \frac{1}{1 + \ln n}$ diverges by the limit Comparsion Test.

(3) $\sum_{n=1}^{\infty} \sin \frac{1}{n}$ $\quad a_n = \sin \frac{1}{n} \quad b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = \lim_{x\to 0} \frac{\sin x}{x} = 1 > 0$$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

Thus, $\sum_{n=1}^{\infty} \sin \frac{1}{n}$ diverges by limit Comparsion Test.

(4) $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{9/2}}$ $\quad a_n = \frac{(\ln n)^2}{n^{9/2}} \quad b_n = \frac{1}{n^{\frac{3}{2}}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{(\ln n)^2}{n^{9/2}}}{1/n^{3/2}} = \lim_{n\to\infty} \frac{(\ln n)^2}{n^3} = \lim_{n\to\infty} \frac{\frac{2\ln n}{n}}{3n^2}$$

$$= \lim_{n\to\infty} \frac{2}{3} \frac{\ln n}{n^3} = \frac{2}{3} \lim_{n\to\infty} \frac{\frac{1}{n}}{3n^2} = \frac{2}{9} \lim_{n\to\infty} \frac{1}{n^3}$$

$$= \frac{2}{9}(0) = 0$$

and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent p-series. Thus, $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{\frac{9}{2}}}$ converges by the limit Comparsion Test.

Exp

What about $\sum_{n=1}^{\infty} \frac{1-n}{n 2^n} = \sum_{n=1}^{\infty} \frac{1}{n 2^n} + \sum_{n=1}^{\infty} \frac{-1}{2^n}$

converges by the Direct Comparsion Test since $\frac{1}{n 2^n} < \frac{1}{2^n}$

Converges since it is a geometric series

Thus, $\sum_{n=1}^{\infty} \frac{1-n}{n 2^n}$ converges since it is sum of two convergent series.

10.5 The Ratio and Root Tests

Th "Ratio Test"

Consider the infinite series $\sum a_n$ with positive terms.

Assume $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$. Then

- if $\rho < 1$, then the series converges.
- if $\rho > 1$, then the series diverges. "or $\rho = \infty$"
- if $\rho = 1$, then the test is inconclusive.

Exp Apply Ratio test to

[1] $\sum_{n=1}^{\infty} \frac{n^2}{e^n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2}{e^{n+1}} \cdot \frac{e^n}{n^2}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 \frac{1}{e} = \frac{1}{e} < 1$$

Thus, the series converges by the ratio test.

---

[2] $\sum_{n=1}^{\infty} \frac{n!}{e^n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!}$$

$$= \lim_{n\to\infty} \frac{n+1}{e} = \infty > 1$$

Thus, the series diverges by the ratio Test.

---

[3] $\sum_{n=1}^{\infty} \frac{1}{n}$ "harmonic series which diverges"

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{n}{1} = 1$$ "Ratio Test is inclusive"

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ "p-series with p=2 whic converges"

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{(n+1)^2} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^2} = 1$$ "Ratio Test is inclusive"

---

[4] $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!\,n!\,3^n}$ converges by Ratio Test $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+4}{3n+3} = \frac{1}{3} < 1$ ✓

Th "The Root Test"

Consider the infinite series $\sum a_n$ with $a_n \geq 0$ for $n \geq N$.

Assume $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$. Then

- if $\rho < 1$, then the series converges
- if $\rho > 1$ or infinit, then the series diverges
- if $\rho = 1$, then the test is inconclusive.

Exp Apply the root test to

$\boxed{1}$ $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$ $\quad \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{n} - \frac{1}{n^2}\right)^n} = \lim_{n\to\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right) = 0 < 1$

Thus, the series converges by the root test.

$\boxed{2}$ $\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$ $\quad \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{\ln n}{n}\right)^n} = \lim_{n\to\infty} \frac{\ln n}{n} = 0 < 1$

Thus, the series converges by the root test.

$\boxed{3}$ $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$ $\quad \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{3^n}{n^3}} = \lim_{n\to\infty} \frac{3}{(\sqrt[n]{n})^3} = \frac{3}{1^3} = 3 > 1$

Thus, the series diverges by the root test.

$\boxed{4}$ $\sum_{n=1}^{\infty} \frac{1}{n}$ "harmonic series which diverges"

$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1$ "Root Test is inconclusive"

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ "p-series which converges $p=2$"

$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{(\sqrt[n]{n})^2} = \frac{1}{1^2} = 1$ "Ratio Test is inconclusive"

Exp Consider the recursive defined terms: $a_1 = 2$, $a_{n+1} = \frac{2}{n} a_n$. Does $\sum_{n=1}^{\infty} a_n$ converg

$a_2 = \frac{2^2}{1!}$, $a_3 = \frac{2^3}{2!}$, $a_4 = \frac{2^4}{3!}$, $a_5 = \frac{2^5}{4!}$, $a_6 = \frac{2^6}{5!}$ $\dots$ $a_n = \frac{2^n}{(n-1)!}$ $\boxed{n \geq 1}$

Apply Ratio Test $\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2}{n} = 0 < 1 \Rightarrow$ The series converges.

10.6 Alternating Series
Absolute and Conditional Convergence

* Alternating Series is a series in which the terms are alternately positive and negative.

Exp [1] Alternating harmonic series:

converge by Th follows

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

[2] Alternating geometric series:

converge geometric

$$-1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \cdots + \frac{(-1)^n}{2^n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} = \frac{-1}{1+\frac{1}{2}} = \frac{-2}{3}$$

[3] Alternating series can diverge:

diverge by th n^{th} term test

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n+1} n + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} n$$

Th "The Alternating Series Test" "Leibniz's Test" Let $u_n = |a_n|$

The alternating Series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots$ converges if

[1] $u_n > 0$ for all n and

[2] $u_{n+1} \leq u_n$ for all $n \geq N$ "nonincreasing" and

[3] $\lim_{n \to \infty} u_n = 0$

Exp The alternating harmonic series is then converges by the alternating Series Test. To prove [2] let $f(x) = \frac{1}{x}$. If $f'(x) \leq 0$ then f is nonicreasing $\forall x > 0 \Leftrightarrow n = 1, 2, \cdots$

* If $u_n = \dfrac{10n}{n^2 + 16}$, for what values of n does the sequence satisfy [2].

Let $f(x) = \dfrac{10x}{x^2 + 16} \Rightarrow f'(x) = \dfrac{10(16 - x^2)}{(x^2 + 16)^2} \leq 0$ whenever $x \geq 4$

Thus, $u_{n+1} \leq u_n$ for $n \geq 4$. That is, the sequence is nonincreasing for $n \geq 4$.

* Assume the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} U_n = U_1 - U_2 + U_3 - U_4 + \cdots$$ converge to L

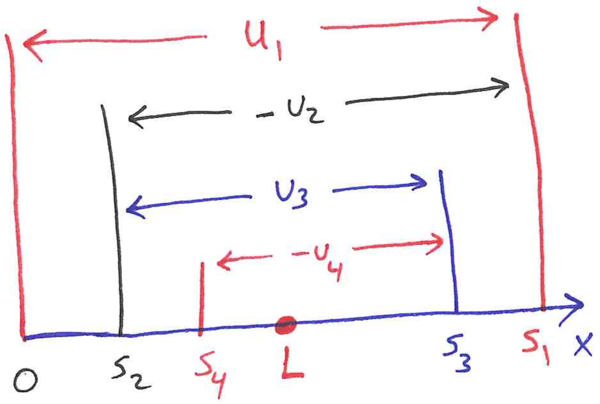
satisfies $\boxed{1}, \boxed{2}, \boxed{3}$

* $S_1 = U_1 > 0$

$S_2 = U_1 - U_2 = S_1 - U_2 > 0$

$S_3 = U_1 - U_2 + U_3 = S_2 + U_3 > 0$

$S_4 = U_1 - U_2 + U_3 - U_4 = S_3 - U_4$



* L lies between any two successive sums S_n and S_{n+1}.

$$S_n < L < S_{n+1}$$

* L differens from S_n by less than U_{n+1}:

$$\underbrace{|L - S_n|}_{\text{Absolute value of the error}} < U_{n+1} \quad \text{for } n \geq N$$

Absolute value of the error "magnitude"
"Remainder"

Exp ② Approximate the sum of $\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!}$ with an error of magnitude less than 5×10^{-6}.

First we find how many terms should be used to estimate the sum with an error $< 5 \times 10^{-6}$

$$\frac{1}{(2n)!} < 5 \times 10^{-6} \Rightarrow (2n)! > \frac{10^6}{5} = 200,000$$

$$\Rightarrow n \geq 5$$

$|\text{error}| < U_6 = |a_6| = \frac{1}{9!}$
$= 0.00000276$
$< 0.000005 = 5 \times 10^{-6}$

2nd $\quad S_5 = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} \approx 0.54030$

Ⓧ goes here...

---

① $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} + \frac{1}{256} - \cdots = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$

* Note that • $0.6640625 = S_8 < L = \frac{2}{3} < S_9 = 0.66796875$

• $\underbrace{L - S_n}_{\text{Remainder "Error"}} = L - S_8 = \frac{2}{3} - 0.6640625 = 0.0026041666$

$< U_9 = \frac{1}{256} = 0.00390625$

Remainder
"Error"

Def A series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges

Exp[1] $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$ converges absolutely because $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a convergent p-series

Exp[2] $\sum_{n=1}^{\infty} (-1)^{n+1} (0.1)^n$ converges absolutely because $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$ which is a convergent geometric series

Exp[3] $\sum_{n=1}^{\infty} (-1)^n \frac{\sin n}{n^2}$ converges absolutely by the Direct Comparasion Test because $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2} \le \sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a convergent p-series.

Def: A series $\sum a_n$ is called converges conditionally if it converges but not absolutely.

Exp $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ "Alternating harmonic series" converges conditionally because $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$ which is the divergent harmonic serie

Exp $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converges by the Alternating Series Test

$\boxed{1}+\boxed{2}$ $u_n = \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} = u_{n+1} > 0$

$\boxed{3}$ $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

But $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ which is a divergent p-series. Thus, $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converges conditionally.

Exp* ** $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$, $p>0$ converges absolutely if $p>1$

converges conditionally if $0 < p \le 1$

Th If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

see Exp[1], Exp[2], Exp[3]

That is if the series converges absolutely, then it converges.

Proof : For each $n \Rightarrow -|a_n| \le a_n \le |a_n|$

$$\Rightarrow 0 \le a_n + |a_n| \le 2|a_n|$$

$$\Rightarrow 0 \le \sum_{n=1}^{\infty}(a_n + |a_n|) \le \underbrace{\boxed{2\sum_{n=1}^{\infty}|a_n|}}_{\text{converges}}$$

Thus, by the Direct Comparasion Test $\sum_{n=1}^{\infty}(a_n + |a_n|)$ converges

Since $a_n = a_n + |a_n| - |a_n|$

$$\Rightarrow \sum_{n=1}^{\infty} a_n = \underbrace{\boxed{\sum_{n=1}^{\infty}(a_n + |a_n|)}}_{\text{converge}} - \underbrace{\boxed{\sum_{n=1}^{\infty}|a_n|}}_{\text{converges}}$$

. Thus $\sum_{n=1}^{\infty} a_n$ converges

Examples ① $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3+n}{5+n}$ diverges by the n^{th} term test

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{3+n}{5+n} = 1 \ne 0$$

---

② $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1+n}{n^2}$ converges conditionally since

[1]+[2] $f(x) = \frac{1+x}{x^2} = \frac{1}{x^2} + \frac{1}{x} \Rightarrow f'(x) = -\left(\frac{2}{x^3} + \frac{1}{x^2}\right) < 0 \Rightarrow$

$f(x)$ is decreasing and so $u_n > u_{n+1} > 0$

[3] $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1+n}{n^2} = \lim_{n\to\infty}\left(\frac{1}{n^2} + \frac{1}{n}\right) = 0$ so it converges

by the alternating series test.

But $\sum_{n=1}^{\infty}|a_n| = \underbrace{\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2}}}_{\text{convergent p-series}} + \underbrace{\boxed{\sum_{n=1}^{\infty}\frac{1}{n}}}_{\text{divergent harmonic series}}$ which diverges.

Th15 (The Alternating Series Estimation Theorem)

If $\sum_{n=1}^{\infty}(-1)^{n+1}u_n = L$, then $s_n = u_1 - u_2 + u_3 - \cdots + (-1)^{n+1}u_n$ approximate L with an error $|L - s_n| < u_{n+1} = |a_{n+1}|$. Furthermore, $s_n < L < s_{n+1}$ and the remainder $L - s_n$ has the same sign as a_{n+1}.

10.7 Power Series

- Power Series are sum of infinite polynomials.

Def • A power series about $x=a$ has the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

where $c_0, c_1, c_2, \ldots, c_n, \ldots$ constant coefficients and a is the center.

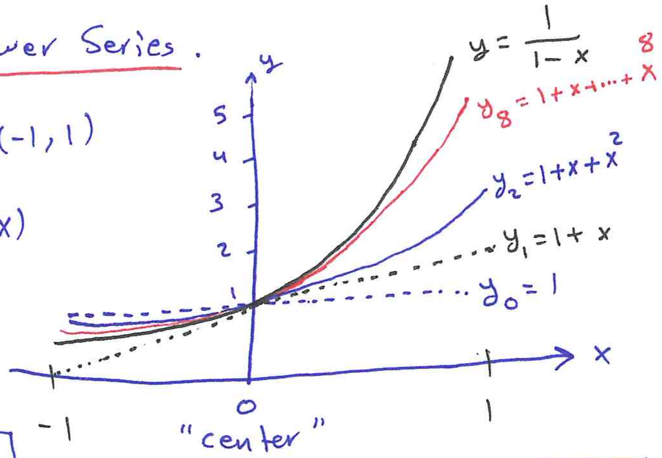
- A power series about $x=0$ is then given by

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots \qquad \cdots\cdots(1)$$

Exp If $c_0 = c_1 = \cdots = c_n = \cdots = 1$ in (1), then we get the geometric power series $\displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}$ if $-1<x<1$

which we call the Reciprocal Power Series.

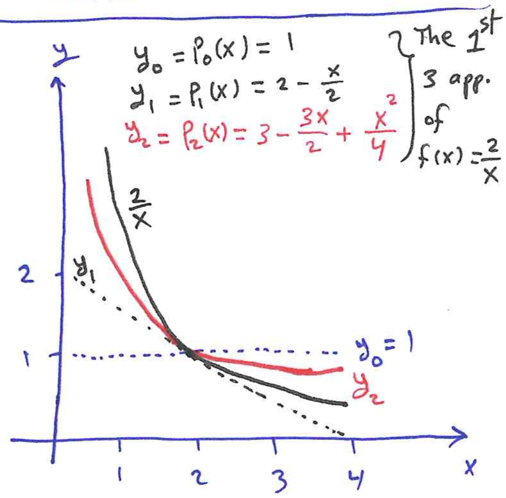
- To approximat $f(x) = \dfrac{1}{1-x}$ on $(-1,1)$ we use the partial sums $y_n = P_n(x)$
- The approximations works only on $(-1,1)$ in which $f(x)$ is continuous.



Exp Consider the power series

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots$$

- The center: $a=2$
- The coefficients: $c_0 = 1$, $c_1 = \frac{-1}{2}$, $c_2 = \frac{1}{4}, \ldots, c_n = \left(\frac{-1}{2}\right)^n$
- The geometric series converges to $\dfrac{1}{1-r}$ where $|r| = \left|\frac{-1}{2}(x-2)\right| < 1 \iff 0<x<4$.
- The sum is $\dfrac{1}{1-r} = \dfrac{1}{1+\frac{x-2}{2}} = \dfrac{2}{x}$
- So, $\dfrac{2}{x} = 1 - \dfrac{(x-2)}{2} + \dfrac{(x-2)^2}{4} + \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots, \quad 0<x<4$



Exp Find the radius of convergence and the interval of convergence for the following power series:

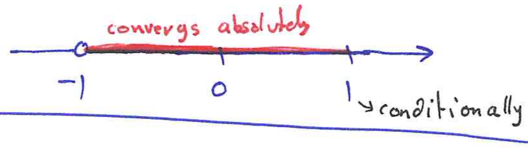
1 $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ Apply Ratio Test to $\sum |u_n|$

- $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n}\right| = |x| \lim_{n\to\infty} \frac{n}{n+1} = |x| < 1$
- Thus, the radius of convergence is $R = 1$.
- To find the interval of convergence: $-1 < x < 1$
  - The series converges absolutely for $-1 < x < 1$
  - The series diverges for $|x| > 1$
  - To check the endpoints:

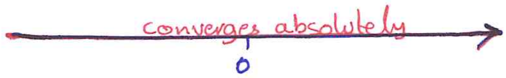
→ converges conditionally • when $x = 1 \Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ "alternating harmonic series which converges"

- when $x = -1 \Rightarrow \sum_{n=1}^{\infty} (-1)^{2n-1} \frac{1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$ "negative of harmonic series which diverges."

- Thus, the series converges for $-1 < x \leq 1$

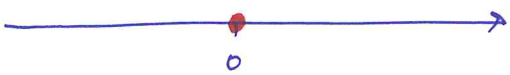


2 $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ Apply Ratio Test

- $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$ for all x
- Thus, $R = \infty$ → • The series converges absolutely for all x.
- 

3 $\sum_{n=0}^{\infty} n!\, x^n$ Apply Ratio Test: $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!\, x^{n+1}}{n!\, x^n}\right|$

$= |x| \lim_{n\to\infty} (n+1) = \infty$ except $x = 0$

- Thus, $R = 0$
- The series diverges for all values of x except $x = 0$
- 

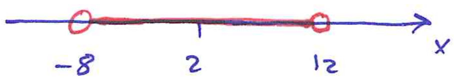
$\boxed{4}$ $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$ Apply Ratio Test $\displaystyle\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{10^{n+1}}\cdot\frac{10^n}{(x-2)^n}\right|$

$$= \frac{|x-2|}{10} < 1$$

- Thus, $R = 10$
- The series converges absolutely for $|x-2| < 10 \iff -8 < x < 12$
- when $x = -8 \Rightarrow \displaystyle\sum_{n=0}^{\infty} (-1)^n$ which diverges since $a_n \not\to 0$ as $n\to\infty$
- when $x = 12 \Longrightarrow \displaystyle\sum_{n=0}^{\infty} 1$ which diverges

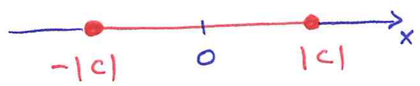
$\boxed{\text{center} = 2}$

- Thus,

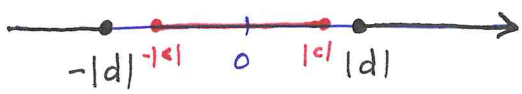


Th Consider the power series $\displaystyle\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$

- If the series converges at $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$



- If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$



Exp $\boxed{2}$ the series converges at $x = 3 \Rightarrow$ it converges absolutely for $|x| < 3 \iff -3 < x < 3$

Exp $\boxed{3}$ the series diverges at $x = 3 \Rightarrow$ it diverges for $|x| > 3$

Exp $\boxed{4}$ the series converges at $x = 3 \Rightarrow$ it converges absolutely for $|x-2| < 3 \iff -1 < x < 5$

Th If $\displaystyle\sum_{n=0}^{\infty} a_n x^n = A(x)$ and $\displaystyle\sum_{n=0}^{\infty} b_n x^n = B(x)$ converge absolutely for $|x| < R$,

then $\displaystyle\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right)$ converges absolutely to $A(x)\,B(x)$ for $|x| < R$:

That is: $\displaystyle A(x)\,B(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$, where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0$$

Th 20 If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely on $|x| < R$, then

$\sum_{n=0}^{\infty} a_n [f(x)]^n$ converges absolutely, on $|f(x)| < R$ for any continuous function f.

Exp $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ converges absolutely for $|x| < 1$

$\Rightarrow \frac{1}{1-4x^2} = \sum_{n=0}^{\infty} (4x^2)^n$ converges absolutely for $|4x^2| < 1 \Leftrightarrow |x| < \frac{1}{2}$

Exp Use Th 20 to find the interval of convergence and the sum of the series as function of x: $\sum_{n=0}^{\infty} \left(\frac{x^2+1}{3}\right)^n$ Apply Ratio Test

- $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left| \frac{(x^2+1)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{(x^2+1)^n} \right| = \frac{|x^2+1|}{3} < 1 \Leftrightarrow x^2 < 2 \Leftrightarrow -\sqrt{2} < x < \sqrt{2}$
- At $x = \pm\sqrt{2} \Rightarrow \sum_{n=0}^{\infty} (1)^n$ which diverges.
- Thus, the interval of convergence is $-\sqrt{2} < x < \sqrt{2}$
- The series $\sum_{n=0}^{\infty} \left(\frac{x^2+1}{3}\right)^n$ is convergent geometric series on $-\sqrt{2}<x<\sqrt{2}$
- The sum is $\frac{1}{1-\frac{x^2+1}{3}} = \frac{3}{2-x^2}$

Th (Term by Term Differentiation)

Assume that $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ converges absolutely on $|x-a| < R$.

Then f has derivatives of all orders on $|x-a| < R$. That is,

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}, \quad f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2}, \quad \ldots$$

converge at every point on $|x-a| < R$.

Exp $f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ on $|x| < 1$ converges absolutely if $|\text{ratio}| < 1$

$f'(x) = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots$ on $|x| < 1$

$f''(x) = \sum_{n=2}^{\infty} n(n-1) x^{n-2} = 2 + 6x + 12x^2 + \cdots$ on $|x| < 1$

Th (Term by Term Integration Theorem)

suppose that $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ converges for $|x-a| < R$.

Then, $\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$ converges for $|x-a| < R$ and $\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$.

Exp Identify the function $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ $\quad |x| \le 1$. $*$

Note that $f'(x) = 1 - x^2 + x^4 - x^6 + \cdots \quad |x| < 1$

$f'(x) = \frac{1}{1+x^2}$ geometric series

$$\boxed{\frac{\pi}{4} = \tan^{-1} 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}}$$

Leibniz's Formula

Hence, $f(x) = \int f'(x)\,dx = \tan^{-1} x + C$.

To find $C \Rightarrow$ From $*$ $f(0) = 0 \iff 0 = \tan^{-1} 0 + C \iff \boxed{C = 0}$

Hence, $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \tan^{-1} x$, $\quad |x| \le 1$

Exp The series $\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$ converges on $|t| < 1$.

Therefore $\ln(1+x) = \int_0^x \frac{dt}{1+t} = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big|_0^x$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

or $\quad \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad |x| < 1$

$$\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

$\boxed{10.8}$ Taylor and Maclaurin Series

Def Let f be a smooth function "all derivatives exist" on an interval that contains the interior point a. Then

* The Taylor series generated by f at $x=a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

* The Maclaurin series generated by f is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

Note that
- the Maclaurin series generated by f is the Taylor series generated by f at $x=0$.
- The function $f(x)$ could be approximated by the Talyor polynomials: $P_0(x), P_1(x), \dots, P_n(x)$

$P_0(x) = f(a)$ Polynomial of degree 0

$P_1(x) = f(a) + f'(a)(x-a)$ "linearization" Polynomial of degree 1

$P_2(x) = P_1(x) + \frac{f''(a)}{2!}(x-a)^2$ Polynomial of degree 2

$P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$ Polynomial of degree 3

$\vdots$

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Polynomial of degree $\underline{\underline{n}}$

Exp 1 Find the Talyor series of $f(x) = e^x$ at $x=0$.

$f(x) = e^x \Rightarrow f(0) = 1$

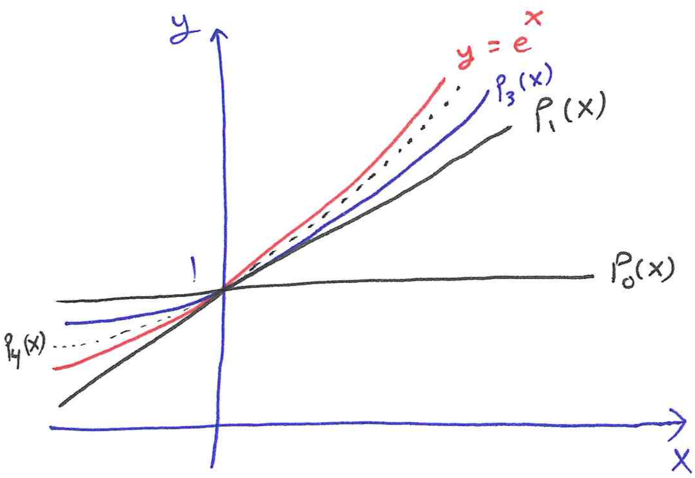
$f'(x) = e^x \Rightarrow f'(0) = 1$

$f''(x) = e^x \Rightarrow f''(0) = 1$

$f'''(x) = e^x \Rightarrow f'''(0) = 1$

$\vdots \qquad \vdots$

$f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = 1$



The Taylor series generated by f at $x=0$ is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots = \boxed{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

$\Downarrow$

This is also The Maclaurin series of e^x.

The series converges to e^x for every x as $n \to \infty$

2 Find Taylor polynomials of order 0, 1, 2, 3

$P_0(x) = 1$

$P_1(x) = 1 + x$

$P_2(x) = 1 + x + \frac{x^2}{2!}$

$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

Exp Find the Talyor series and Taylor polynomials generated by $f(x) = \cos x$ at $x=0$

$f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f^{(4)}(x) = \cos x$

$f^{(5)}(x) = -\sin x$

$\vdots$

$f^{(2n)}(x) = (-1)^n \cos x$

$f^{(2n+1)}(x) = (-1)^{n+1} \sin x$

$f^{(2n)}(0) = (-1)^n$

$f^{(2n+1)}(0) = 0$

The Talyor series generated by f at $x=0$ is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$= 1 + 0 \cdot x - \frac{x^2}{2!} + 0 \cdot x^3 + \frac{x^4}{4!} + \cdots$$

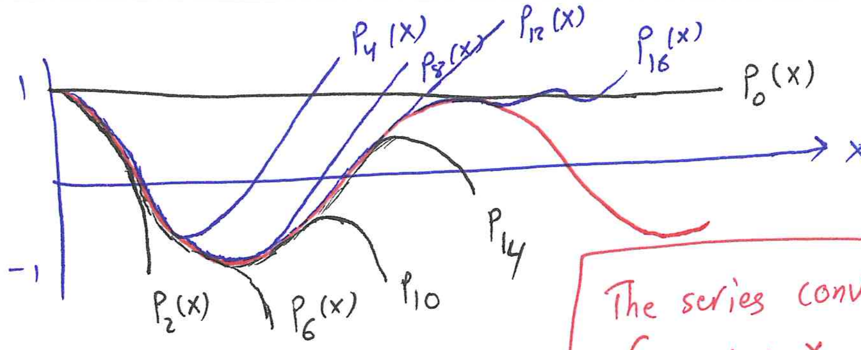
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$ even function

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

This is also the Maclaurin series of $\cos x$.

Talyor Polynomial of order $2n$ is

$$P_{2n}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}$$



$P_0(x) = 1$

$P_2(x) = 1 - \frac{x^2}{2!}$

The series converges to $\cos x$ for every x as $n \to \infty$

Exp Find Taylor series generated by $f(x) = \sin x$ at $x = 0$.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

converges for every x to $\sin x$ as $n \to \infty$.

Exp Find the Maclaurin series for the function $\cosh x$.

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left[e^x + e^{-x}\right]$$

$$= \frac{1}{2}\left[1 + \cancel{x} + \frac{x^2}{2!} + \cancel{\frac{x^3}{3!}} + \cdots + 1 - \cancel{x} + \frac{x^2}{2!} - \cancel{\frac{x^3}{3!}} + \cdots\right]$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

Exp Find the Maclaurin series for the function $f(x) = \begin{cases} 0, & x = 0 \\ e^{-1/x^2}, & x \neq 0 \end{cases}$

The Maclaurin series of $f(x)$ is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$= 0 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + \cdots$$

$$= 0 + 0 + 0 + 0 + 0 + \cdots$$

$$= 0$$

The series converges for every x but converges to $f(x)$ only at $x = 0$

Thus, the Taylor series generated by $f(x)$ does not converge to $f(x)$ as $n \to \infty$.

$\boxed{10.9}$ Convergence of Taylor Series

Th (Taylor's Theorem)

Assume $f, f', f'', \dots, f^{(n)}$ are continous on $[a,b]$ and $f^{(n)}$ is differentiable on (a,b), then there exists a number $c \in (a,b)$ such that:

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

* Note that Taylor's Theorem is a generalization of the MVT.
* If we change b by x, we get Taylor's formula:

$$f(x) = \underbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{P_n(x)} + R_n(x)$$

where

- $R_n(x)$ is the Remainder of order n "or the error term" results from approximating f by $P_n(x)$ on an open interval I contains a given by

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{where } c \in (a,x)$$

* Convergence of Taylor Series:

If $\lim_{n\to\infty} R_n(x) = 0$ for all $x \in I$, then the Taylor Series generated by f at $x = a$ converges to f on I:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

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Exp Show that the Taylor series generated by $f(x)=e^x$ at $x=0$ converges to $f(x)$ for every real value x.

$f(x)=e^x$ is smooth on $\mathbb{R}=(-\infty,\infty)$. Thus, has all orders of all derivatives. Taylor's formula is

$c\in(0,x)$

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+R_n(x) \quad \text{, where}$$

$R_n(x)=\frac{e^c}{(n+1)!}x^{n+1}$. Note that $\lim\limits_{n\to\infty} R_n(x) = e^c \lim\limits_{n\to\infty}\frac{x^{n+1}}{(n+1)!}=0$

Thus, the series converges to e^x for every $x \Longrightarrow$ Th 5 Sec. 10.1

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^k}{k!}+\cdots=\sum_{k=0}^{\infty}\frac{x^k}{k!}$$

---

* Note that $e=\sum\limits_{k=0}^{\infty}\frac{1}{k!}$ as a series: $c\in(0,1)$ $0<c<1$, $e^0<e^c<e<3$

$$e = 1+1+\frac{1}{2!}+\cdots+\frac{1}{n!}+R_n(1), \text{ where } R_n(1)=\frac{e^c}{(n+1)!}<\frac{3}{(n+1)!}$$

Th (The Remainder Estimation Theorem)

Given Taylor's formula: $f(x)=P_n(x)+R_n(x)$.

If $|f^{(n+1)}(t)|\le M$ for all $t\in(a,x)$, then the Remainder

$$\boxed{|R_n(x)|\le\frac{M}{(n+1)!}|x-a|^{n+1}.}\ *$$

Further, if * holds for every n, then the series converges to $f(x)$.

Exp Show that the Taylor series for $\sin x$ at $x=0$ converges for all x. $c\in(a,x)=(0,x)$

Recall that Taylor's formula for $\sin x$ at $x=0$ is

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+R_{2n+1}(x).$$

Now apply The Remainder Estimation Theorem:

$$|R_{2n+1}(x)|=\left|\frac{f(c)}{(2n+2)!}x^{2n+2}\right|\le\frac{|x|^{2n+2}}{(2n+2)!}.$$

But $\lim\limits_{n\to\infty}|R_{2n+1}(x)|\le\lim\limits_{n\to\infty}\frac{|x|^{2n+2}}{(2n+2)!}=0 \quad \forall x$

$\Rightarrow$ The Maclaurin series for $\sin x$ converges to $\sin x$ $\forall x$.

$\Rightarrow \sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots=\sum\limits_{0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$.

Exp show that Taylor series for $\cos x$ at $x=0$ converges to $\cos x$ for every value of x.

The Taylor's formula for $\cos x$ at $x=0$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + R_{2n}(x).$$

Since $|\cos x|$ and $|$it's all derivative$| < 1$ we apply the Remainder Estimation Theorem with $\underline{M=1}$

$|R_{2n}(x)| \leq \frac{|x|^{2n+1}}{(2n+1)!}$. Now for every x we have $R_{2n}(x) \to 0$ as $n \to \infty$ by Th5 section 10.1

Therefore, the series converges to $\cos x$ for every x. Thus,

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Exp Find the first four nonzero terms in the Maclaurin series for the functions:

[1] $\frac{1}{3}(2x + x\cos x) = \frac{2x}{3} + \frac{x}{3}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$

$$= \frac{2x}{3} + \frac{x}{3} - \frac{x^3}{2!\,3} + \frac{x^5}{3(4!)} - \frac{x^7}{3(6!)} + \cdots$$

$$= x - \frac{x^3}{6} + \frac{x^5}{72} - \frac{x^7}{2160} + \cdots$$

[2] $e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$

$$= \left(\underline{\underline{1+x}} + \cancel{\frac{x^2}{2!}} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - \left(\cancel{\frac{x^2}{2!}} + \frac{x^3}{2!} + \frac{x^4}{2!\,2!} + \frac{x^5}{2!\,3!} + \cdots\right)$$

$$+ \left(\frac{x^4}{4!} + \frac{x^5}{4!} + \frac{x^6}{2!\,4!} + \cdots\right)$$

$$= \underline{\underline{1 + x}} - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

---

[3] $\cos 2x$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!} = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots$$

$$= 1 - \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} - \frac{2^6 x^6}{6!} + \cdots$$

Exp For what values of x can we replace $\sin x$ by $x - \frac{x^3}{3!}$ with an error of magnitude no more than 3×10^{-4}?

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

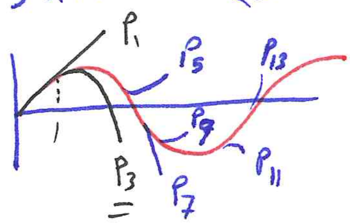
Using the Alternating Series Estimation Theorem:

The error after $\frac{x^3}{3!} < \left| \frac{x^5}{5!} \right|$. Therefore the error

$u_{n+1} = |a_{n+1}|$

will be less than 3×10^{-4} $\iff$ $\frac{|x|^5}{120} < 3 \times 10^{-4}$ $\iff$

$$|x| < \sqrt[5]{360 \times 10^{-4}} \simeq 0.514.$$



Exp Estimate the error if $P_3(x) = x - \frac{x^3}{6}$ is used to estimate the value of $\sin x$ at $x = 0.1$

$f(x) = \sin x = P_3(x) + R_3(x)$, where $R_3(x) = \frac{f^{(4)}(c)}{4!}(x-0)^4 = \frac{f^{(4)}(c)}{4!} x^4$

Apply The Remainder Estimation Th. with $M = 1$

$M \le |f^{(4)}(c)| \le 1$

$$\text{Error} = |R_3(x)| \le \frac{|x^4|}{4!} = \frac{(0.1)^4}{24} < 4.2 \times 10^{-6}$$

$\boxed{10.10}$ The Binomial Series and Applications of Taylor Series

* The Binomial series of $f(x) = (1+x)^m$ is "using Taylor series expa."

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k \quad , \quad \underbrace{|x| < 1}_{\text{where the series converges absolutely.}} \quad , \quad m \text{ is constant}$$

where $\binom{m}{1} = m \quad , \quad \binom{m}{2} = \dfrac{m(m-1)}{2}$

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!} \quad \text{for } k \geq 3$$

Powers and Roots

Exp Find the first four terms of the binomial series for

$\boxed{1}$ $$(1+x)^{\frac{1}{2}} = 1 + \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} x^k = 1 + \frac{x}{2} + \frac{(\frac{1}{2})(-\frac{1}{2})x^2}{2} + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})x^3}{3!} + \cdots$$
$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots$$

$\boxed{2}$ $$\left(1 - \frac{x}{2}\right)^{-2} = 1 + \sum_{k=1}^{\infty} \binom{-2}{k} \left(\frac{-x}{2}\right)^k$$
$$= 1 + x + \frac{(-2)(-3)\left(\frac{-x}{2}\right)^2}{2} + \frac{(-2)(-3)(-4)\left(-\frac{x}{2}\right)^3}{3!} + \cdots$$
$$= 1 + x + \frac{3x^2}{4} + \frac{x^3}{2} + \cdots$$

Approximating Nonelementary Integrals

Exp Use series to estimate the following integrals with an error of magnitude less than 10^{-3}

$\boxed{1}$ $$\int_0^{0.1} \frac{dx}{\sqrt{1+x^4}} = \int_0^{0.1} (1+x^4)^{-\frac{1}{2}} dx = \int_0^{0.1} \left(1 + \sum_{k=1}^{\infty} \binom{-\frac{1}{2}}{k} x^{4k}\right) dx$$
$$= \int_0^{0.1} \left(1 - \frac{x^4}{2} + \frac{3x^8}{8} - \cdots\right) dx = \left[x - \frac{x^5}{10} + \frac{3x^9}{9(8)} - \cdots\right]_0^{0.1}$$
$$\approx x \Big|_0^{0.1} \approx 0.1 \text{ with error } |E| \leq \frac{(0.1)^5}{10} \approx 0.000001 .$$

$\boxed{2}$ $\displaystyle\int_0^{0.2} \frac{e^{-x}-1}{x}\,dx = \int_0^{0.2} \frac{1}{x}\left(e^{-x}-1\right)dx$ $\qquad e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$= \int_0^{0.2} \frac{1}{x}\left(\cancel{1} - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots \cancel{-1}\right)dx$$

$$= \int_0^{0.2} \left(-1 + \frac{x}{2!} - \frac{x^2}{3!} + \frac{x^3}{4!} - \cdots\right)dx$$

$$= \left[-x + \frac{x^2}{4} - \frac{x^3}{18} + \frac{x^4}{4(4!)} - \cdots\right]_0^{0.2} \approx -(0.2) + \frac{(0.2)^2}{4} - \frac{(0.2)^3}{18} \approx -0.19044$$

with error $|E| \le \frac{(0.2)^4}{96} \approx 0.00002$

Arctangents: Remember that

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

section 10.7

with Leibniz's formula:

$$\frac{\pi}{4} = \tan^{-1}1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots + \frac{(-1)^n}{2n+1} + \cdots$$

- Note that $\tan^{-1}x = \displaystyle\int \frac{dx}{1+x^2}$

$$\Rightarrow \frac{1}{1+x^2} = \frac{d}{dx}\tan^{-1}x = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \frac{(-1)^{n+1}x^{2n+2}}{1+x^2}$$

- Integrate both sides from 0 to x $\Rightarrow$

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \underbrace{\int_0^x \frac{(-1)^{n+1}t^{2n+2}}{1+t^2}\,dt}_{R_n(x)}$$

where $\displaystyle|R_n(x)| \le \int_0^{|x|} t^{2n+2}\,dt = \frac{|x|^{2n+3}}{2n+3} \xrightarrow[n\to\infty]{} 0$ as $n\to\infty$ if $|x|\le 1$

- If $|x| \le 1$, then $\displaystyle\lim_{n\to\infty} R_n(x) = 0$ and so

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \quad |x| \le 1.$$

Indeterminate forms

Exp Use series to evaluate the limits :

1. $\lim\limits_{x\to 1} \dfrac{\ln x}{x-1}$

Taylor Series of $\ln x$ about $x=1$ is

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \cdots$$

$$= \lim_{x\to 1}\left(1 - \frac{1}{2}(x-1) + \cdots\right) = 1$$

2. $$\lim_{x\to 0} \frac{e^x - (1+x)}{x^2} = \lim_{x\to 0} \frac{1}{x^2}\left(e^x - 1 - x\right)$$

$$= \lim_{x\to 0} \frac{1}{x^2}\left(\cancel{1} + \cancel{x} + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \cancel{-1} \cancel{-x}\right)$$

$$= \lim_{x\to 0}\left(\frac{1}{2} + \frac{x}{3!} + \cdots\right) = \frac{1}{2}$$

3. $$\lim_{y\to 0} \frac{y - \tan^{-1} y}{y^3} = \lim_{y\to 0} \frac{1}{y^3}\left[\cancel{y} - \left(\cancel{y} - \frac{y^3}{3} + \frac{y^5}{5} - \cdots\right)\right]$$

$$= \lim_{y\to 0}\left[\frac{1}{3} - \frac{y^2}{5} + \cdots\right] = \frac{1}{3}$$

Euler's formula

$i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$

Recall that $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!} + \dfrac{x^6}{6!} + \cdots$

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \frac{i^6\theta^6}{6!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \to \text{Euler's formula}$$

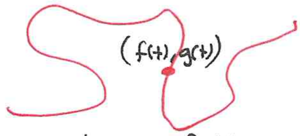
θ is polar angle.

- Any complex number has the form $a + bi$, $a, b \in \mathbb{R}$.

Exp $e^{i\pi} = \cos\pi + i\sin\pi = -1$

11.1 Parametrization of Plane Curves

* We may describe the movement of a particle in the xy plane at position t by $(x(t), y(t)) = (f(t), g(t))$



position of the particle at time t
not a function

Def If x and y are given as functions

$$x = f(t) \quad \text{and} \quad y = g(t) \quad , \ t \in I,$$

then the set of points $(x, y) = (f(t), g(t))$ is a parameteric curve.

Note that 1. $x = f(t)$ and $y = g(t)$ are called parametric equations.

2. the variable t is called the parameter of the curve.

3. the interval I is called the parameter interval.

$\Rightarrow$ If $I = [a, b]$ closed interval, then the point $(f(a), g(a))$ is the initial point and the point $(f(b), g(b))$ is the terminal point.

4. We say that we have parametrized the curve, if we find 1 and 3. That is 1 and 3 give a parametrization of the curve.

Exp Given the parametric equation and parameter interval:

$$x = t^2 \quad , \quad y = t + 1 \quad , \quad -\infty < t < \infty$$

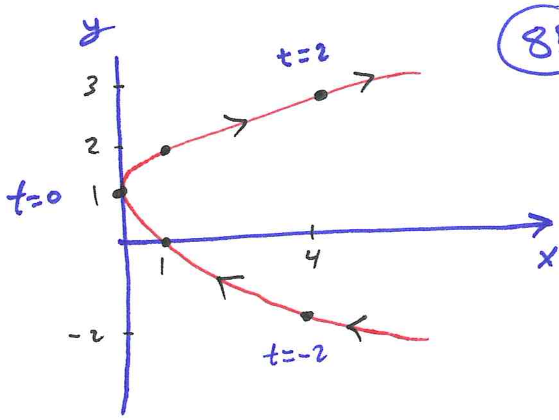
1. Find the Cartesian (→ algebraic) equation by eliminating the parameter t
2. Idenitify the particle's path by sketching the cartesian equation
3. Find the direction of motion

1. Cartesian equation: $x = t^2 = (y-1)^2 \Leftrightarrow x = (y-1)^2$

Note that sometimes it's difficult or even impossible to eliminate the parameter t.

The curve that represents the particle movement.

t	-3	-2	-1	0	1	2	3
x	9	4	1	0	1	4	9
y	-2	-1	0	1	2	3	4

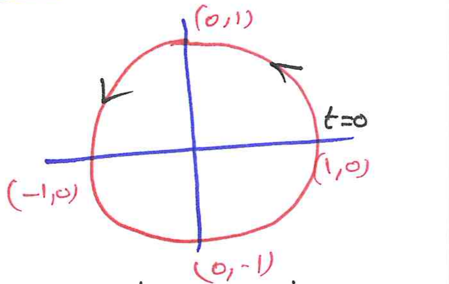


Exp Graph the parametric curve of $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$

- we can eliminate the parameter t by:

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1 \iff \boxed{x^2 + y^2 = 1} \text{ cartesian equation}$$

- Initial point is $(\cos 0, \sin 0) = (1, 0)$
- Terminal point is $(\cos 2\pi, \sin 2\pi) = (1, 0)$
- $t = \pi \Rightarrow$ the position is $(-1, 0)$



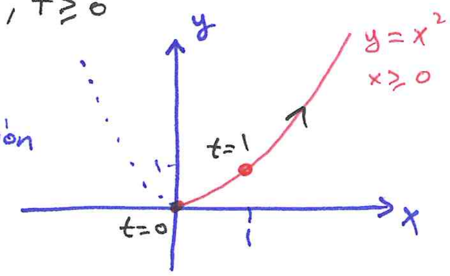
Direction: counterclockwise

Exp Graph the particle's movement and direction if its parametric equation and parameter interval is [1] $x = \sqrt{t}$, $y = t$, $t \ge 0$

- we can eliminate the parameter t

$$y = t = x^2 \iff \boxed{y = x^2} \text{ Cartesian equation}$$

- Initial point is $(\sqrt{0}, 0) = (0, 0)$
- No terminal point
- $t = 1 \Rightarrow$ the position is $(1, 1)$

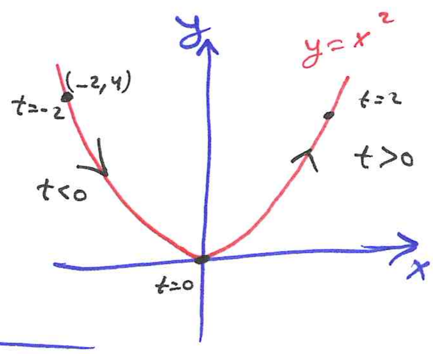


[2] $x = t$, $y = t^2$, $-\infty < t < \infty$

- we can eliminate the parameter t

$$y = t^2 = x^2 \iff \boxed{y = x^2} \to \text{Cartesian equation}$$

- no initial point
- no terminal point



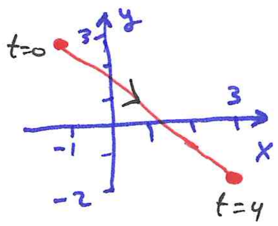
Exp Find a parameterization for the line passes throw the points (a,b) and (c,d).

- A cartesian equation is $y-b=m(x-a)$ where the slope $m=\dfrac{d-b}{c-a}$, $c\neq a$
- Set the parameter $t=x-a$
- Hence, $x=a+t$, $y=b+mt$, $-\infty<t<\infty$ parameterizes the line.

the line segment with endpoints $(-1,3)$ and $(3,-2)$ $\quad m=\dfrac{-2-3}{3+1}=\dfrac{-5}{4}$

$$\left\{x=-1+t,\quad y=3-\frac{5}{4}t,\quad 0\le t\le 4\right.$$

$$\{x=-1+4t,\quad y=3-5t,\quad 0\le t\le 1\}$$



Both parametrization give the same segment

Exp sketch and identify the path by the point $P(x,y)$ if

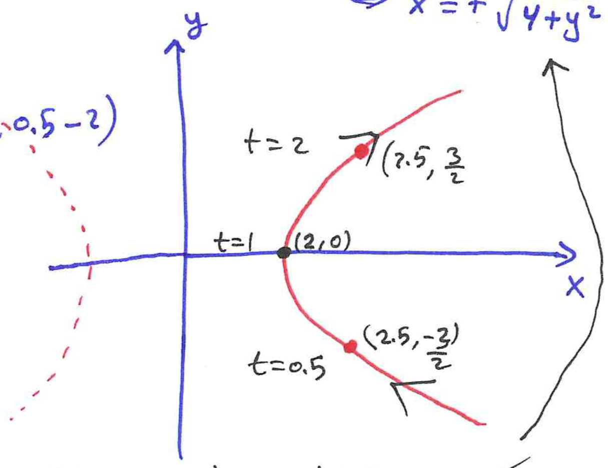
$$x=t+\frac{1}{t},\quad y=t-\frac{1}{t},\quad t>0$$

- We can eliminate the parameter t by:

$$\left.\begin{array}{l} x+y=2t \\ x-y=\dfrac{2}{t}\end{array}\right\}\to (x+y)(x-y)=4 \iff x^2-y^2=4 \iff x^2=4+y^2$$

$$\iff x=+\sqrt{4+y^2}$$

- at $t=0.5 \Rightarrow$ the position is $(0.5+2,\ 0.5-2)$
  $\Rightarrow (2.5, -\frac{3}{2})$
- at $t=2 \Rightarrow$ the position is $(2.5, \frac{3}{2})$



$x>0$ since $t>0$

Note that

$$\left\{x=t+\frac{1}{t},\ y=t-\frac{1}{t},\ t>0\right\}$$

$$\{x=\sqrt{4+t^2},\ y=t,\ -\infty<t<\infty\}$$

$$\left\{x=2\sec t,\ y=2\tan t,\ -\frac{\pi}{2}<t<\frac{\pi}{2}\right\}$$

are all different parametrization for the same curve.

11.2 Calculus with Parametric Curves

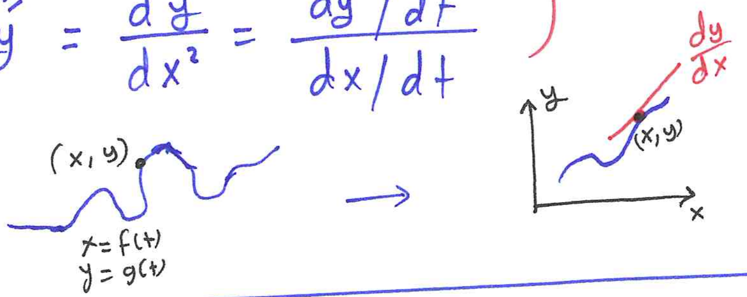
* Parametric formulars for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$:

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$\Big\}$ *

- Given the parametric equations: $x = f(t)$, $y = g(t)$.
- If x, y, y' are differentiable at t with $\frac{dx}{dt} \neq 0$, then * holds at any point t.



Exp • Find the tangent to the curve $x = t$, $y = \sqrt{t}$, $t = \frac{1}{4}$

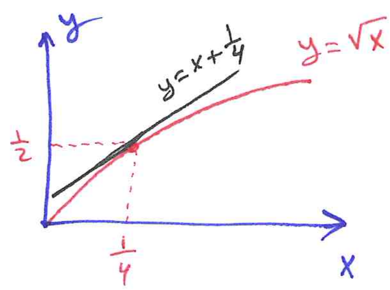
- The point is $\left(f(\tfrac{1}{4}), g(\tfrac{1}{4})\right) = \left(\tfrac{1}{4}, \tfrac{1}{2}\right)$
- The tangent line is $y - \frac{1}{2} = m(x - \frac{1}{4})$, where

the slope $m = \left.\frac{dy}{dx}\right|_{t=\frac{1}{4}} = \left.\frac{dy/dt}{dx/dt}\right|_{t=\frac{1}{4}} = \left.\frac{\frac{1}{2\sqrt{t}}}{1}\right|_{t=\frac{1}{4}} = 1$

$\Rightarrow$ The tangent line becomes $y - \frac{1}{2} = x - \frac{1}{4}$

$$y = x + \frac{1}{4}$$

- Find $\frac{d^2y}{dx^2}$ as a function of t



$$\frac{dy}{dx} = y' = \frac{dy/dt}{dx/dt} = \frac{1}{2\sqrt{t}}$$

$$\frac{d^2y}{dx^2} = y'' = \frac{dy'/dt}{dx/dt} = \frac{\frac{1}{2}(-\frac{1}{2})\, t^{-\frac{3}{2}}}{1}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{4\sqrt{t^3}}$$

- Find $\left.\frac{d^2y}{dx^2}\right|_{t=\frac{1}{4}} = \frac{-1}{4\sqrt{\frac{1}{64}}} = \frac{-1}{4\,\frac{1}{8}} = \frac{-1}{\frac{1}{2}} = -2$

Exp Find the slope of the curve "x and y are implicitly differentiable"

$$x^3 + 2t^2 = 9 \quad , \quad 2y^3 - 3t^2 = 4 \quad \underline{\underline{at}} \quad t = 2$$

- Note that when $\boxed{t=2}$ $\Rightarrow x^3 + 2(2)^2 = 9 \Rightarrow x^3 + 8 = 9 \Rightarrow x^3 = 1 \Rightarrow \boxed{x = 1}$

$\Rightarrow 2y^3 - 3(2)^2 = 4 \Rightarrow 2y^3 - 12 = 4 \Rightarrow 2y^3 = 16 \Rightarrow \boxed{y = 2}$

- The slope is:

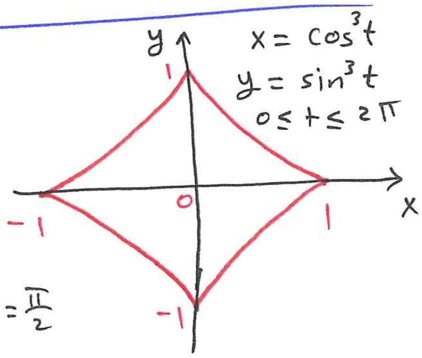
$$m = \left.\frac{dy/dt}{dx/dt}\right|_{t=2}$$

- $3x^2 \frac{dx}{dt} + 4t = 0 \Rightarrow \frac{dx}{dt} = \frac{-4t}{3x^2}$
- $6y^2 \frac{dy}{dt} - 6t = 0 \Rightarrow \frac{dy}{dt} = \frac{t}{y^2}$

$$= \left.\frac{\frac{t}{y^2}}{\frac{-4t}{3x^2}}\right|_{t=2}$$

$$= \frac{\frac{2}{4}}{\frac{-8}{3}} = -\frac{1}{2} \cdot \frac{3}{8} = \frac{-3}{16}$$

Exp Find the area enclosed by the astroid



$x = \cos^3 t$
$y = \sin^3 t$
$0 \le t \le 2\pi$

$$A = 4\int_0^1 y\, dx = 4\int_{\frac{\pi}{2}}^{0} \sin^3 t \; 3\cos^2 t(-\sin t)\, dt$$

when $x = 0 \Rightarrow 0 = \cos^3 t \Rightarrow 0 = \cos t \Rightarrow t = \cos^{-1} 0 = \frac{\pi}{2}$

$x = 1 \Rightarrow 1 = \cos^3 t \Rightarrow 1 = \cos t \Rightarrow t = \cos^{-1} 1 = 0$

$$A = 12\int_0^{\frac{\pi}{2}} \sin^4 t \cos^2 t\, dt = 12\int_0^{\frac{\pi}{2}} \left(\frac{1-\cos 2t}{2}\right)^2 \left(\frac{1+\cos 2t}{2}\right) dt$$

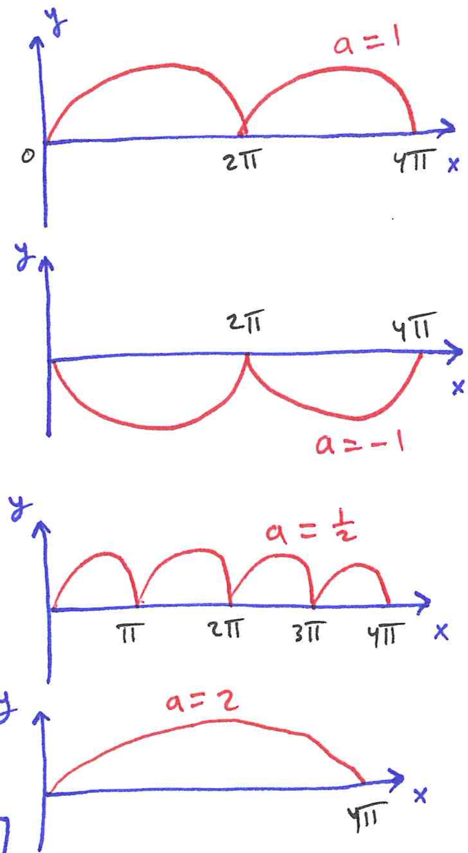
$$= \frac{12}{8}\int_0^{\frac{\pi}{2}} (1 - 2\cos 2t + \cos^2 2t)(1 + \cos 2t)\, dt$$

$$= \frac{3}{2}\int_0^{\frac{\pi}{2}} (1 - \cos 2t - \cos^2 2t + \cos^3 2t)\, dt = \dots = \frac{3\pi}{8}.$$

Exp Find the area under one arch of the cycloid:

$$x = a(t - \sin t), \quad y = a(1 - \cos t)$$

when $a = 1$.



$$A = \int_0^{2\pi} y\, dx = \int_0^{2\pi} (1-\cos t)(1-\cos t)\, dt$$

$$= \int_0^{2\pi} (1-\cos t)^2\, dt = \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\, dt$$

$$= \int_0^{2\pi} \left(1 - 2\cos t + \frac{1+\cos 2t}{2}\right) dt$$

$$= 3\pi$$

Exp Find the area enclosed by the y-axis and the curve $x = t - t^2,\ y = 1 + e^{-t}$, $0 \le t \le 1$

$$A = \int_0^1 x\, dy = \int_0^1 (t - t^2)(-e^{-t})\, dt = \int_1^0 (t-t^2)\, e^{-t}\, dt$$

$(t-t^2)$	$\overset{(+)}{\searrow}$	e^{-t}
$1-2t$	$\overset{(-)}{\searrow}$	$-e^{-t}$
-2	$\overset{(+)}{\searrow}$	e^{-t}
0		$-e^{-t}$

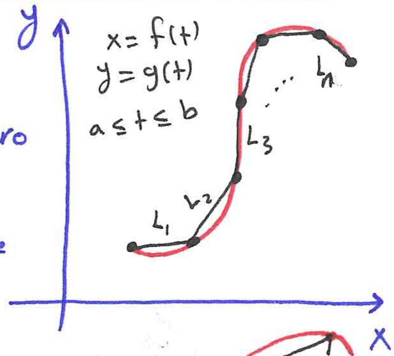
$$= \Big[(t^2 - t)\, e^{-t} - (1-2t)\, e^{-t} + 2 e^{-t}\Big]_1^0 = (-1+2) - (e^{-1} + 2e^{-1})$$

$$= 1 - \frac{3}{e}$$

* **length of a Curve defined parametrically:**

If • f' and g' are continous and not simultaneously zero on $[a,b]$

• the curve traversed exactly once as t increase on $[a,b]$



then the length of the curve is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

$$\sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$L = \lim_{\|P\| \to 0} \sum_{k=1}^{n} L_k$$

Exp Find the length of the curve:

$$x = \frac{t^2}{2}, \quad y = \frac{(2t+1)^{\frac{3}{2}}}{3}, \quad 0 \le t \le 4$$

$$L = \int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^4 \sqrt{t^2 + 2t + 1}\, dt = \int_0^4 \sqrt{(t+1)^2}\, dt$$

$$= \int_0^4 |t+1|\, dt = \int_0^4 (t+1)\, dt = \left(\frac{t^2}{2} + t\right)\Bigg|_0^4 = 12$$

Area of Surfaces of Revolution

Given a smooth curve $x = f(t)$, $y = g(t)$, $a \le t \le b$ traversed exactly once as t increases from a to b.

The area of the surfaces generated by revolving the curve about coordinates axes are as follows:

* Revolution about x-axis $(y \ge 0)$: $S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

* Revolution about y-axis $(x \ge 0)$: $S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

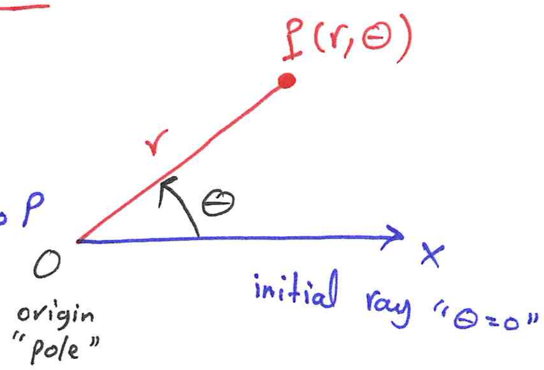
Exp Find the area of surface generating by revolving the curves

- $x = \cos t$, $y = 2 + \sin t$, $0 \le t \le 2\pi$, x-axis

$$S = \int_0^{2\pi} 2\pi (2 + \sin t) \sqrt{\sin^2 t + \cos^2 t}\, dt = 2\pi \int_0^{2\pi} (2 + \sin t)\, dt$$

$$= 2\pi \left[2t - \cos t\right]\Big|_0^{2\pi} = 2\pi\left[4\pi - 1 - (0 - 1)\right] = 8\pi^2$$

$\boxed{11.3}$ Polar Coordinates



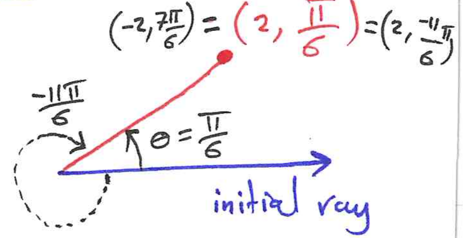
$P(r, \theta)$ is Polar coordinate

- r is the directed distance from O to P
  "r can be negative"
- θ is the directed angle from the initial ray to OP.

Polar coordinates are not unique:

$$P(r, \theta) = P(r, \theta + 2\pi m) \quad , \quad m = 0, \pm 1, \pm 2, \cdots$$

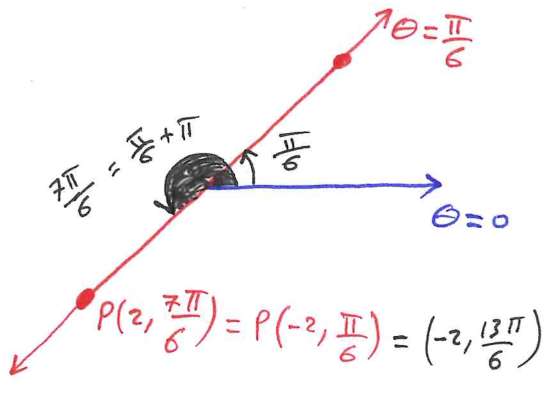
$$= P(-r, (\theta \pm \pi) + 2\pi m) = P(-r, \theta \pm (2m+1)\pi)$$



Exp Polar coordinate $P(2, \frac{\pi}{6})$

$$P(2, \tfrac{\pi}{6}) = (2, \tfrac{\pi}{6} - 2\pi) = (2, -\tfrac{11\pi}{6})$$

$$= (-2, \tfrac{\pi}{6} + \pi) = (-2, \tfrac{7\pi}{6})$$



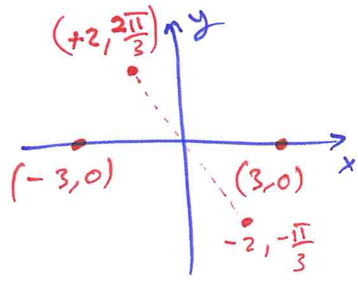
$$P(-2, \tfrac{\pi}{6}) = (-2, \tfrac{\pi}{6} + 2\pi) = (-2, \tfrac{13\pi}{6})$$

$$= (2, \tfrac{\pi}{6} + \pi) = (2, \tfrac{7\pi}{6})$$

Exp The following polar coordinates pair are the same:

$P(r, \theta) = P(3, 0) = P(-3, \pi)$, $(-3, 0) = (-3, 2\pi)$,

$(2, \frac{2\pi}{3}) = (-2, -\frac{\pi}{3})$ ...



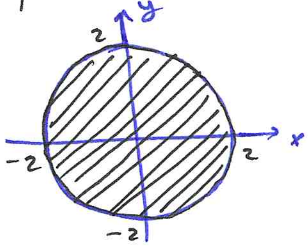
Polar Equations and Graphs:

$r = a$ circle of radius $|a|$ centered at O

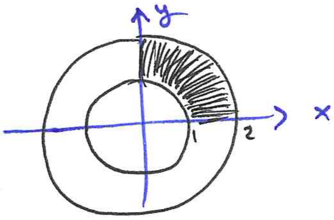
$\theta = \theta_0$ Line through O making an angle θ_0 with the initial ray.

Exp Graph the set of points whose polar coordinates satisfy:

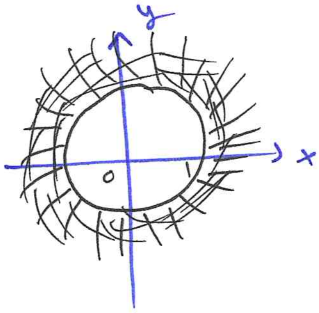
(a) $0 \le r \le 2$



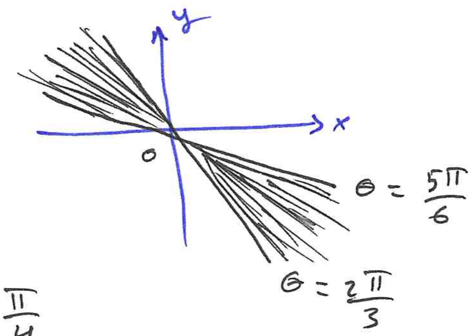
(b) $1 \le r \le 2$ and $0 \le \theta \le \frac{\pi}{2}$



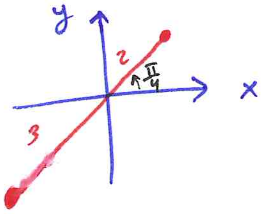
(c) $r \ge 1$



(d) $\frac{2\pi}{3} \le \theta \le \frac{5\pi}{6}$



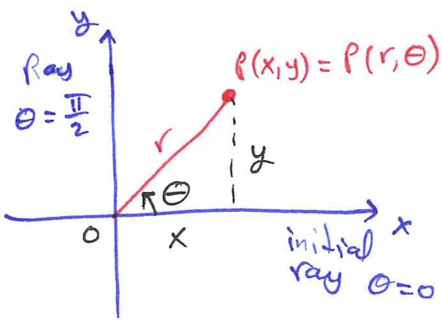
(e) $-3 \le r \le 2$ and $\theta = \frac{\pi}{4}$



Equations Relating Polar and Cartesian Coordinates:

$x = r\cos\theta$, $y = r\sin\theta$

$r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$



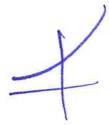
Exp Replace the polar equation with Cartisian equations:

a) $r\cos\theta = 2 \Rightarrow x = 2$ Vertical line passes $(2,0)$

b) $r = -3\sec\theta \Rightarrow r = \dfrac{-3}{\cos\theta} \Rightarrow r\cos\theta = -3 \Leftrightarrow x = -3$

c) $r^2 = 1 \Rightarrow x^2 + y^2 = 1$ circle center $(0,0)$ and $r = 1$

d) $r = \csc\theta \, e^{r\cos\theta} \Rightarrow r\sin\theta = e^{r\cos\theta} \Rightarrow y = e^x$



Exp Replace the cartisian equation with equivalent polar eq.

a) $x = y \Rightarrow r\cos\theta = r\sin\theta \Rightarrow \cos\theta = \sin\theta \Rightarrow \theta = \frac{\pi}{4}$

b) $x^2 + y^2 = 4 \Rightarrow r^2 = 4 \Leftrightarrow r = \pm 2$

c) $(x-5)^2 + y^2 = 25$

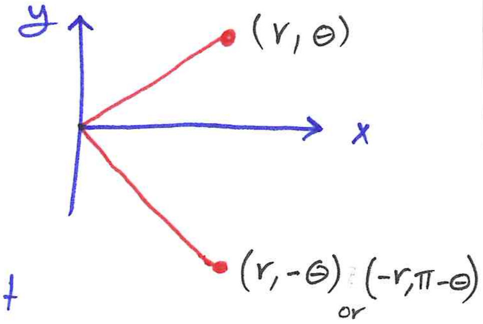
$x^2 - 10x + \cancel{25} + y^2 = \cancel{25} \Leftrightarrow x^2 + y^2 = 10x$

$\Leftrightarrow r^2 = 10 r\cos\theta \Leftrightarrow r = 10\cos\theta$

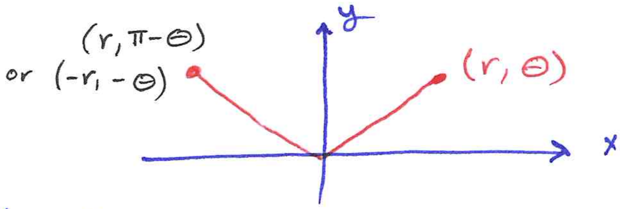
11.4 Graphing in Polar Coordinates

Symmetry Tests for Polar Graphs:

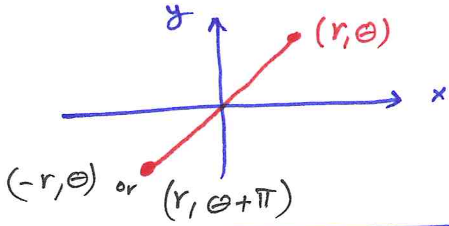
1. **Symmetry about x-axis**: If the point (r,θ) lies on the graph, then $(r,-\theta)$ or $(-r,\pi-\theta)$ lies on the graph.



2. **Symmetry about y-axis**: If the point (r,θ) lies on the graph, then the point $(r,\pi-\theta)$ or $(-r,-\theta)$ lies on the graph



3. **Symmetry about the origin**: If the point (r,θ) lies on the graph, then the point $(-r,\theta)$ or $(r,\theta+\pi)$ lies on the graph.



Slope Let $r=f(\theta)$. Recall the parametric equations:

$$x = r\cos\theta = f(\theta)\cos\theta \qquad , \qquad y = r\sin\theta = f(\theta)\sin\theta \qquad\qquad r' = f'(\theta)$$

slope of the curve $r=f(\theta)$ at (r,θ) is

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$$

Proof $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{f'\sin\theta + f\cos\theta}{f'\cos\theta - f\sin\theta}$

Note that when the curve $r=f(\theta)$ passes through the origin at $\theta_0 \Rightarrow \left.\dfrac{dy}{dx}\right|_{(0,\theta_0)} = \tan\theta_0$

Exp Find the slope of $r = \cos 2\theta$ at $\theta = 0, \frac{\pi}{2}$

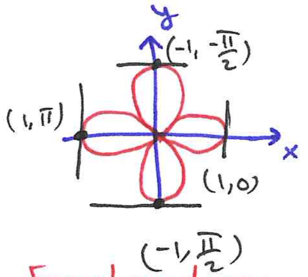
$r' = -2\sin 2\theta$

- when $\theta = 0 \Rightarrow r = 1 \Rightarrow (r, \theta) = (1, 0)$

slope is $\left.\frac{dy}{dx}\right|_{(1,0)} = \left.\frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}\right|_{(1,0)} = \frac{-2\sin(0)\sin(0) + (1)\cos(0)}{-2\sin(0)\cos(0) - (1)\sin(0)} = \frac{1}{0}$ undefined

- when $\theta = \frac{\pi}{2} \Rightarrow r = -1 \Rightarrow (r, \theta) = (-1, \frac{\pi}{2})$

the slope is $\left.\frac{dy}{dx}\right|_{(-1,\frac{\pi}{2})} = \frac{-2\sin(\pi)\sin(\frac{\pi}{2}) + (-1)\cos(\frac{\pi}{2})}{-2\sin(\pi)\cos(\frac{\pi}{2}) - (-1)\sin(\frac{\pi}{2})} = 0$



Four-leaved rose

Exp sketch the grave of the following curves, Identify the symmetry

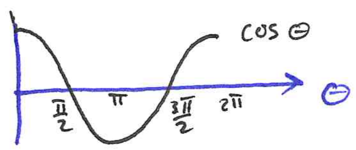
1) $r = 1 - \cos\theta$

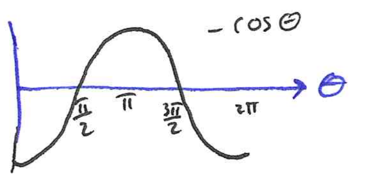
- (r, θ) on the graph $\Rightarrow r = 1 - \cos\theta$
$\Rightarrow r = 1 - \cos(-\theta)$
$\Rightarrow (r, -\theta)$ on the graph

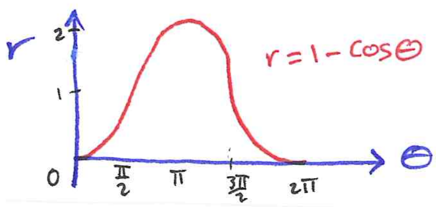
$\Rightarrow$ the curve is symmetric about x-axis.

- $1 - \cos(-\theta) = 1 - \cos\theta \neq -r$
$1 - \cos(\pi - \theta) = 1 + \cos\theta \neq r$
$\Rightarrow$ the curve is not symmetric about y-axis

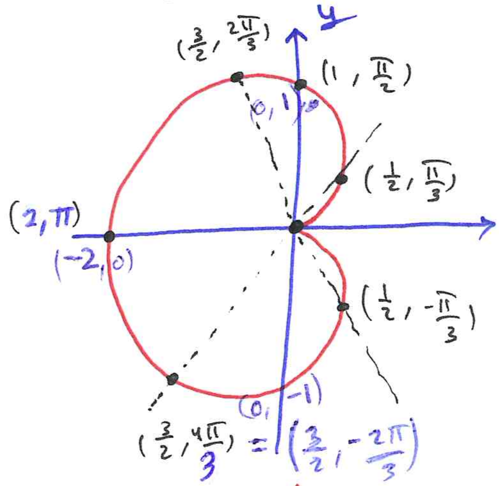
- $1 - \cos\theta \neq -r$
$1 - \cos(\theta + \pi) = 1 + \cos\theta \neq r$
$\Rightarrow$ the curve is not symmetric about the origin.







θ	$r = 1 - \cos\theta$
0	0
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2



cardioid

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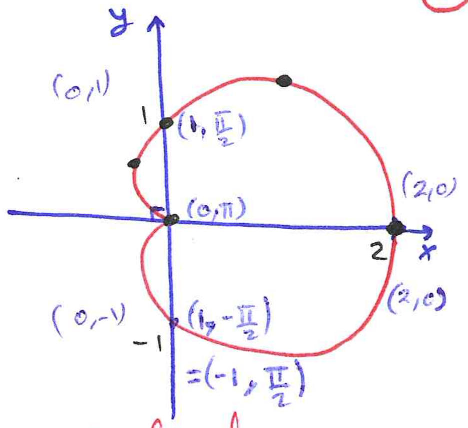
2 $r = 1 + \cos\theta$ ✓

- (r,θ) on the graph $\Rightarrow$

  $r = 1+\cos\theta \Rightarrow r = 1+\cos(-\theta) \Rightarrow$

  $(r,-\theta)$ on the graph $\Rightarrow$ symmetric about x-axis



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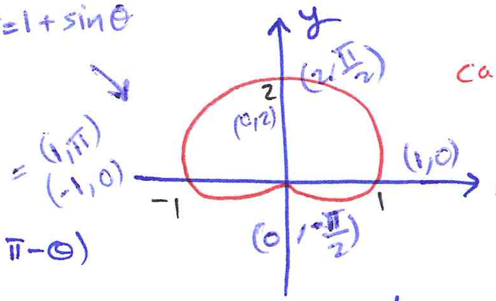
3 $r = -1 + \sin\theta$ similar to $r = 1+\sin\theta$ ✓

- (r,θ) on the graph $\Rightarrow$

  $r = -1 + \sin\theta \Rightarrow r = -1 + \sin(\pi-\theta)$

  $\Rightarrow (r, \pi-\theta)$ on the graph $\Rightarrow$ the curve is symmetric about y-axis



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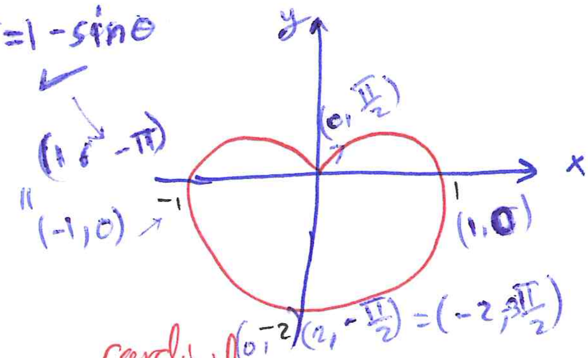
- $\left.\begin{array}{l} -1+\sin(-\theta) = -1-\sin\theta \neq r \\ -1+\sin(\pi-\theta) = -1+\sin\theta \neq -r \end{array}\right\} \Rightarrow$ not symmetric about x-axis.

- $\left.\begin{array}{l} -1+\sin\theta \neq -r \\ -1+\sin(\theta+\pi) = -1-\sin\theta \neq r \end{array}\right\} \Rightarrow$ not symmetric about origin.

---

4 $r = -1 - \sin\theta$ similar to $r = 1-\sin\theta$ ✓

symmetric about y-axis



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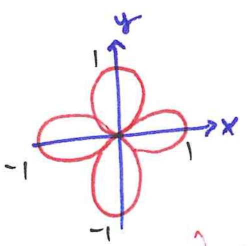
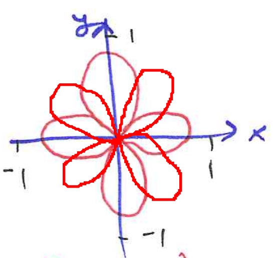
5 $r = \cos 2\theta$



Four-leaved rose

symmetric about x-axis and y-axis and origin

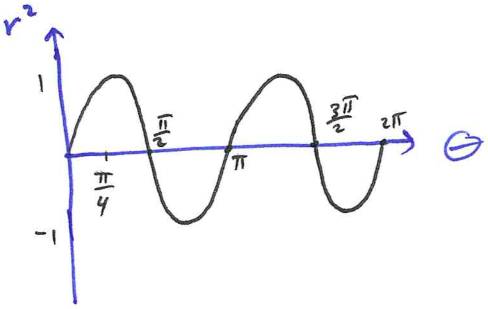
6 $r = \sin 2\theta$

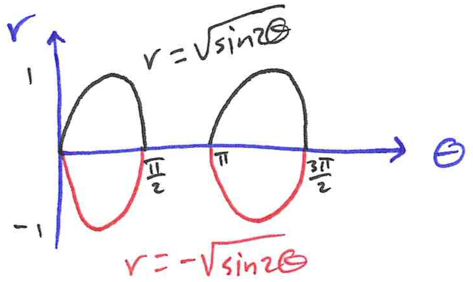


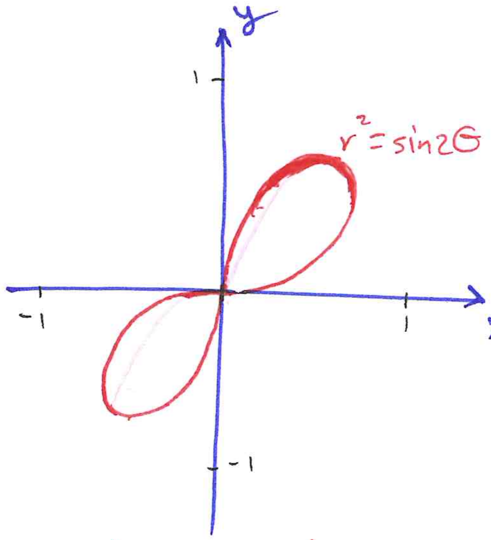
four leaved rose

symmetric about x-axis, y-axis and origin.

$\boxed{7}$ $r^2 = \sin 2\theta$

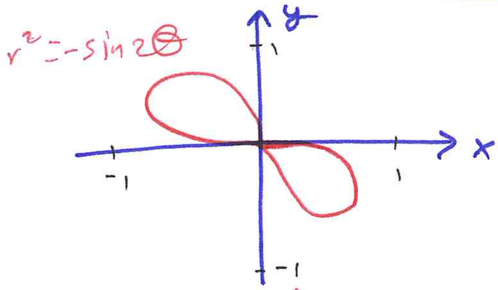






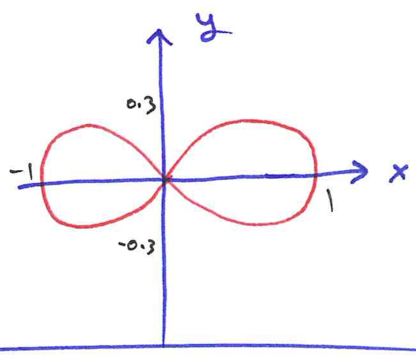
lemniscate

$\boxed{7}$ $r^2 = -\sin 2\theta$

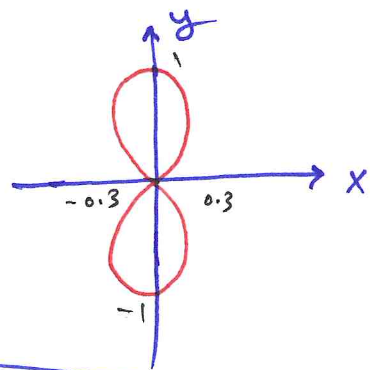


Lemniscate

$\boxed{8}$ $r^2 = \cos 2\theta$

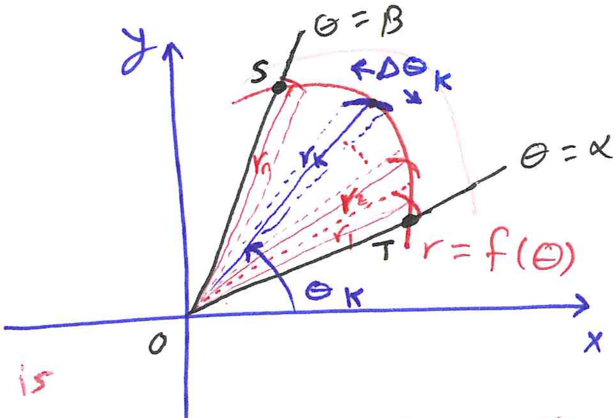


$\boxed{9}$ $r^2 = -\cos 2\theta$



11.5 Areas and lengths in Polar Coordinates

- The region OTS is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$.

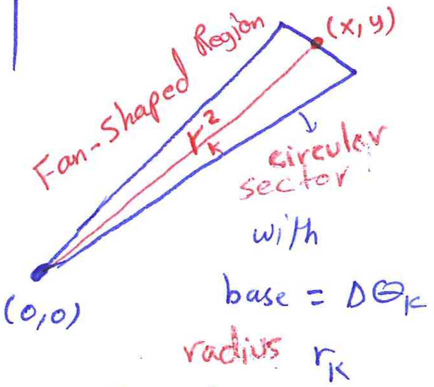


- Area of the circular sector k is

$$A_K = \frac{1}{2} r_K^2 \, \Delta\theta_K$$

$$= \frac{1}{2}\left[f(\theta_k)\right]^2 \Delta\theta_K$$

$A = \pi r^2 = \pi r^2 \frac{\theta}{2\pi} = \frac{r^2\theta}{2}$



- The area of the region can be approximated by

$$\tilde{A} = \sum_{k=1}^{n} A_K = \sum_{k=1}^{n} \frac{1}{2}\left[f(\theta_k)\right]^2 \Delta\theta_K$$

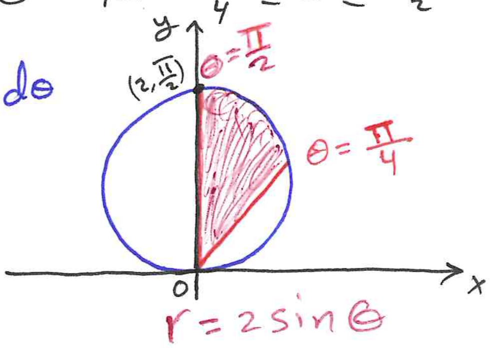
- The approximation is improved as $\|P\| \to 0$ "$n \to \infty$"
- $$\boxed{A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta} = \lim_{\|P\| \to 0} \tilde{A}$$

Exp Find the areas of the region

1. bounded by the circle $r = 2\sin\theta$ for $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$ $\quad (\equiv x^2 + (y-1)^2 = 1)$

$$A = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}\left[2\sin\theta\right]^2 d\theta = 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 - \cos 2\theta)\, d\theta = \frac{\pi}{4} + \frac{1}{2}$$

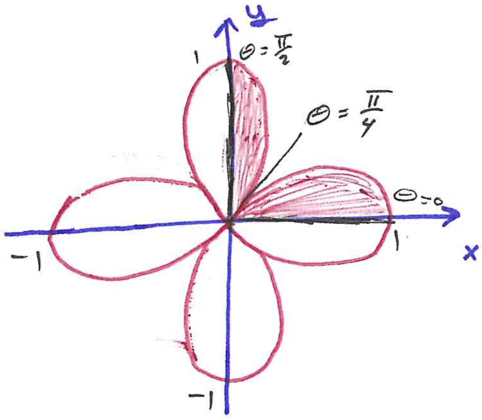


[2] Inside one leaf of the four-leaved rose $r = \cos 2\theta$

$$A = 2\int_0^{\frac{\pi}{4}} \frac{1}{2}[\cos 2\theta]^2 d\theta$$

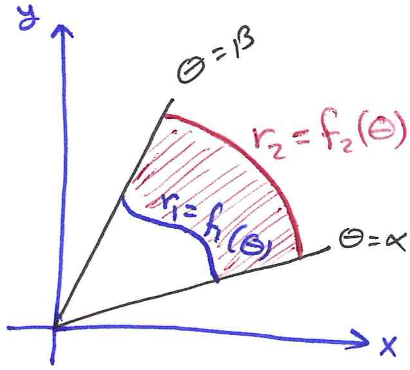
$$= \int_0^{\frac{\pi}{4}} \frac{1+\cos 4\theta}{2} d\theta = \frac{\pi}{8}$$

$$= 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}[\cos 2\theta]^2 d\theta$$



$$A = \int_\alpha^\beta \frac{1}{2} r_2^2 d\theta - \int_\alpha^\beta \frac{1}{2} r_1^2 d\theta$$

$$= \int_\alpha^\beta \frac{1}{2}(r_2^2 - r_1^2) d\theta$$



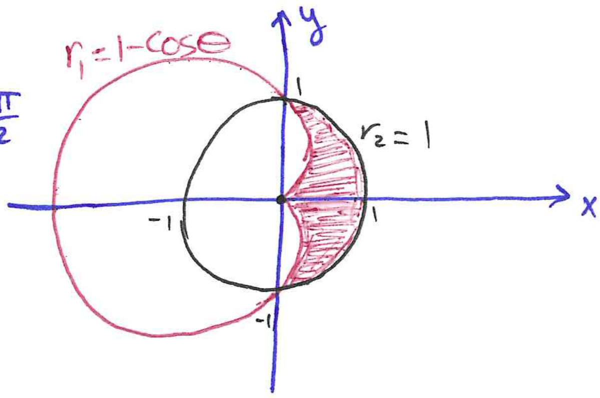
__Exp__ Find the area that lies inside the circle $r=1$ and outside the cardioid $r = 1-\cos\theta$

$r = 1 = 1 - \cos\theta \iff \cos\theta = 0 \iff \theta = \frac{\pi}{2}, -\frac{\pi}{2}$

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(r_2^2 - r_1^2) d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}[1^2 - (1-\cos\theta)^2] d\theta = 2 - \frac{\pi}{4}$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{1}{2}[1^2 - (1-\cos\theta)^2] d\theta = \int_0^{\frac{\pi}{2}} [1 - (1-\cos\theta)^2] d\theta = 2 - \frac{\pi}{4}$$



Length of a Polar Curve

- Recall that $x = r\cos\theta = f(\theta)\cos\theta$
  $y = r\sin\theta = f(\theta)\sin\theta$ $\quad \alpha \le \theta \le \beta$

are parametrization for the curve $r = f(\theta)$.

- The length of r is $L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

$$\frac{dx}{d\theta} = -r\sin\theta + \cos\theta \frac{dr}{d\theta}$$

$$\frac{dy}{d\theta} = r\cos\theta + \sin\theta \frac{dr}{d\theta}$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2\sin^2\theta + \cos^2\theta\left(\frac{dr}{d\theta}\right)^2 - \cancel{2r\sin\theta\cos\theta\frac{dr}{d\theta}} + r^2\cos^2\theta + \sin^2\theta\left(\frac{dr}{d\theta}\right)^2 + \cancel{2r\sin\theta\cos\theta\frac{dr}{d\theta}}}$$

$$\boxed{L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$

<u>Exp</u> Find the length of the curve given by the spiral

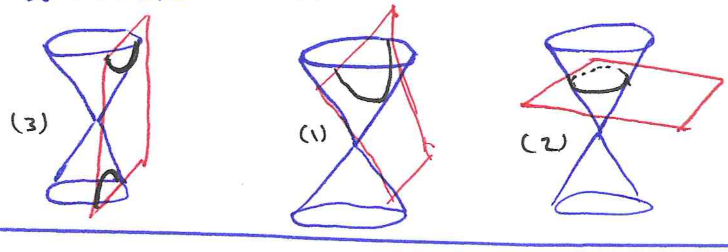
$r = \frac{e^{\theta}}{\sqrt{2}}$, $0 \le \theta \le \pi$

$$L = \int_0^{\pi} \sqrt{\frac{e^{2\theta}}{2} + \frac{e^{2\theta}}{2}}\, d\theta = \int_0^{\pi} \sqrt{e^{2\theta}}\, d\theta = \int_0^{\pi} e^{\theta}\, d\theta$$

$$= e^{\pi} - 1$$

11.6 Conic Sections

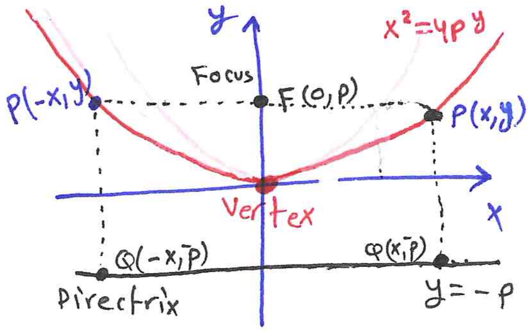
* Conic Sections are Parabolas (1), Ellipses (2), Hyperbolas (3) curves in which a plane cuts a double cone.



Parabolas :

[1] The standard form of the parabola is

$$\boxed{x^2 = 4py \;,\; p>0}$$



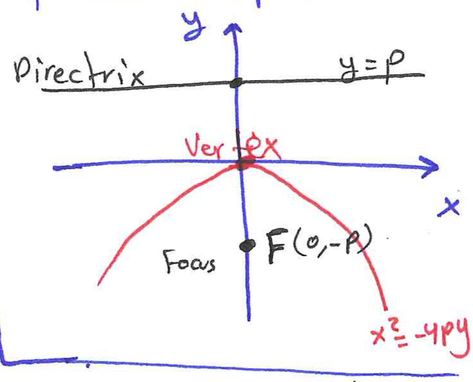
• the Parabola is the set of points s.t $PF = PQ \iff$

$$\sqrt{(x-0)^2+(y-p)^2} = \sqrt{(x-x)^2+(y--p)^2}$$

$$\sqrt{x^2+(y-p)^2} = \sqrt{(y+p)^2} \iff x^2+(y-p)^2 = (y+p)^2 \iff \boxed{x^2=4py}$$

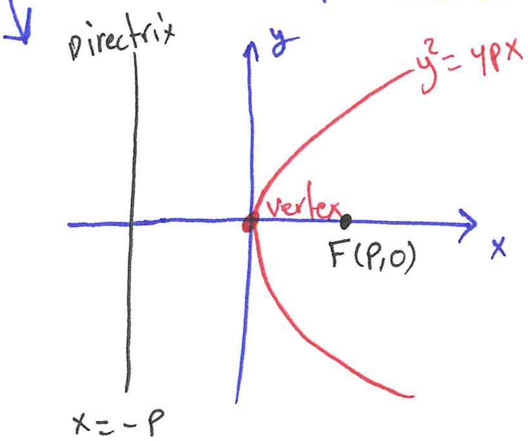
• p is positive number called the parabola's focal length.

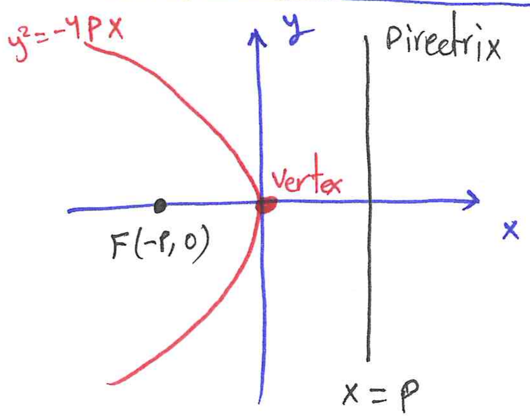
[2] when $x^2 = -4py$, $p>0$ ⇒ the parabola opens down:



[3] when $y^2 = 4px$, $p>0$ ⇒ the parabola opens right

[4] when $y^2 = -4px$, $p>0$ ⇒ the parabola opens left



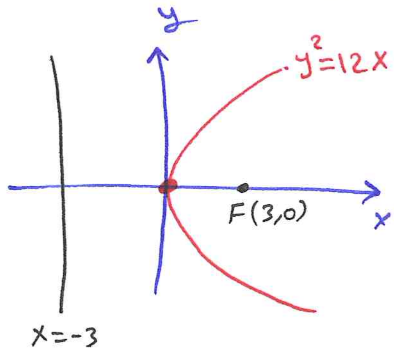


Exp Find the focus and directrix for each of the following parabolas. sketch each one.

[1] $y^2 = 12x$

$4p = 12 \Leftrightarrow p = 3$, vertex is $(0,0)$

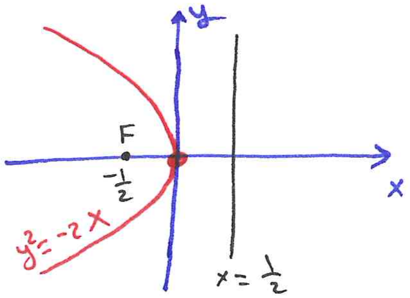
Focus is $(3,0)$, Directrix is $x = -3$



[2] $y^2 = -2x$

$4p = 2 \Leftrightarrow p = \frac{1}{2}$, vertex is $(0,0)$

Focus is $(-\frac{1}{2},0)$, Directrix is $x = \frac{1}{2}$



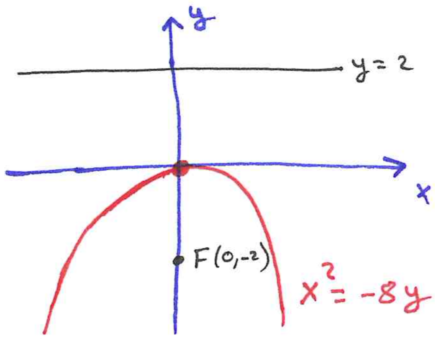
[3] $x^2 = -8y$

$4p = 8 \Leftrightarrow p = 2$

Focus is $(0,-2)$

Directrix is $y = 2$

Vertex is $(0,0)$



[4] $y = 4x^2$

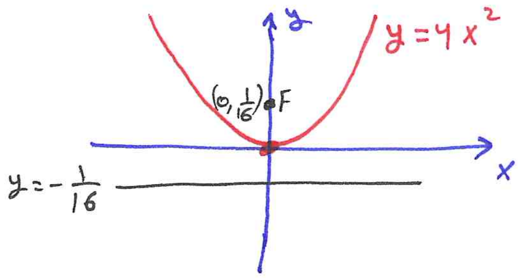
$x^2 = \frac{1}{4}y$

$4p = \frac{1}{4} \Leftrightarrow p = \frac{1}{16}$

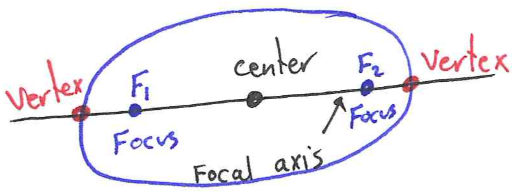
Focus is $(0, \frac{1}{16})$

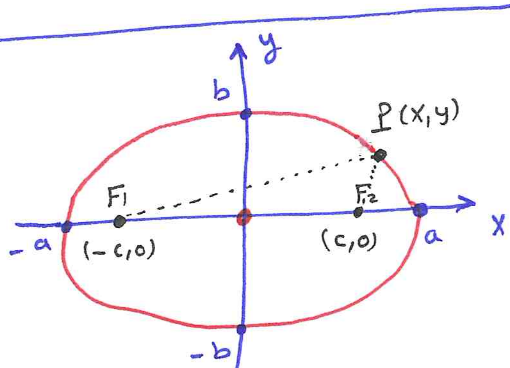
Directrix is $y = -\frac{1}{16}$

Vertex is $(0,0)$



Ellipse





$\boxed{a > b}$ *

The ellipse defined by the equation $PF_1 + PF_2 = 2a$ is the graph of

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $c^2 = a^2 - b^2$ is the Center-to-focus distance.

$a \longrightarrow$ is the semimajor axis

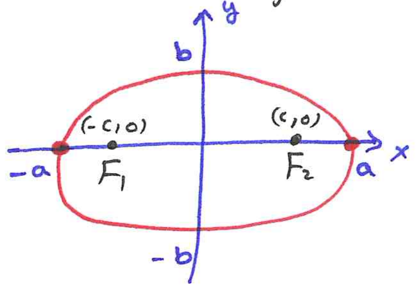
$b \longrightarrow$ is the semiminor axis

* • Major axis of the ellipse is the line segment of length $2a$

* • Minor axis of the ellipse is the line segment of length $2b$.

Standard-Form Equations for Ellipse Centered at the Origin:

* Foci on the x-axis: $\boxed{a > b}$ $\qquad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



- Center-to-focus distance $c = \sqrt{a^2 - b^2}$
- Foci: $(\pm c, 0)$
- Vertices: $(\pm a, 0)$

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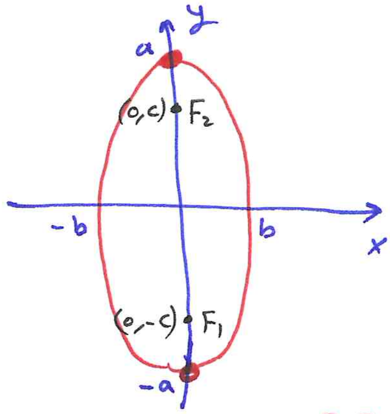
* Foci on the y-axis: $\boxed{a > b}$ $\qquad \dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$



- Center-to-focus distance $c = \sqrt{a^2 - b^2}$
- Foci: $(0, \pm c)$
- Vertices: $(0, \pm a)$

if $a = b \Rightarrow c = 0 \Rightarrow$ we get a circle

Exp Put each of the following equations in the standard form. Sketch the ellipse and include the foci.

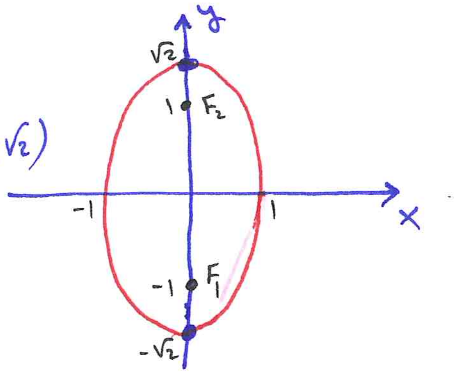
1. $2x^2 + y^2 = 2$

$$\boxed{x^2 + \frac{y^2}{2} = 1}$$

$$c = \sqrt{a^2 - b^2} = \sqrt{2 - 1} = 1$$

Vertices: $(0, \pm\sqrt{2})$

Foci: $(0, \pm 1)$



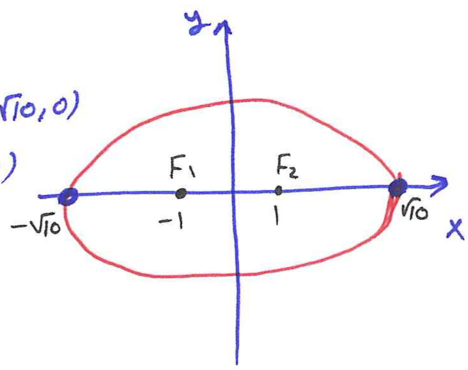
2. $9x^2 + 10y^2 = 90$

$$\boxed{\frac{x^2}{10} + \frac{y^2}{9} = 1}$$

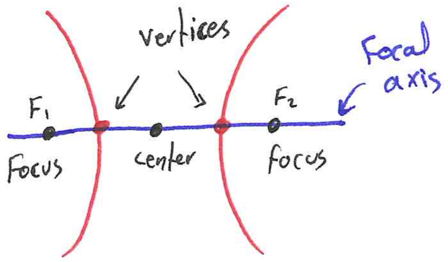
$$c = \sqrt{a^2 - b^2} = \sqrt{10 - 9} = 1$$

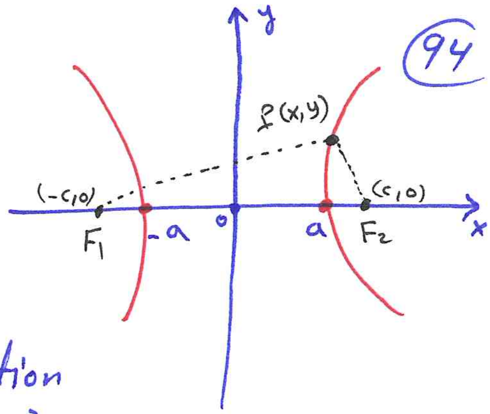
Vertices $(\pm\sqrt{10}, 0)$

Foci: $(\pm 1, 0)$



Hyperbolas



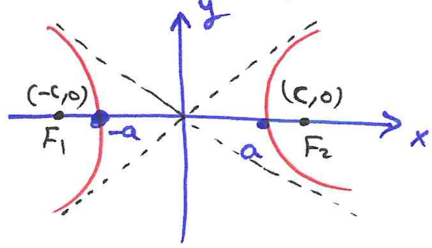


The hyperbola defined by the equation

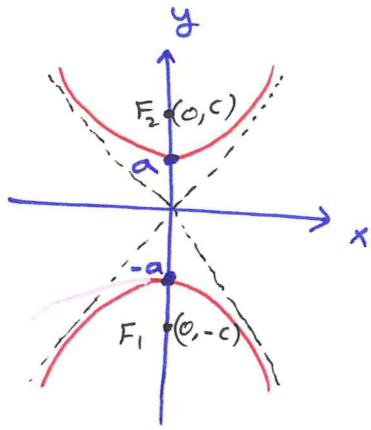
$PF_1 - PF_2 = 2a$ is the graph of $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ where $c = \sqrt{a^2+b^2}$

Standard form Equations for Hyperbolas Centered at Origin:

* Foci on the x-axis: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$
  - Center-to-focus distance $c = \sqrt{a^2+b^2}$
  - Foci: $(\pm c, 0)$
  - Vertices: $(\pm a, 0)$
  - Asymptotes: $y = \pm \dfrac{b}{a} x$



* Foci on the y-axis: $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$
  - Center-to-focus distance $c = \sqrt{a^2+b^2}$
  - Foci: $(0, \pm c)$ Vertices: $(0, \pm a)$
  - Asymptotes: $y = \pm \dfrac{a}{b} x$

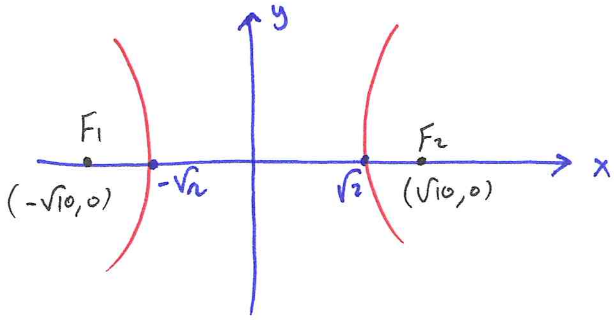


Exp Put each of the following equations in the standard form. Sketch the hyperbola and include the foci and asymptotes.

1. $8x^2 - 2y^2 = 16 \Leftrightarrow \dfrac{x^2}{2} - \dfrac{y^2}{8} = 1$ $a = \sqrt{2}$, $b = 2\sqrt{2}$

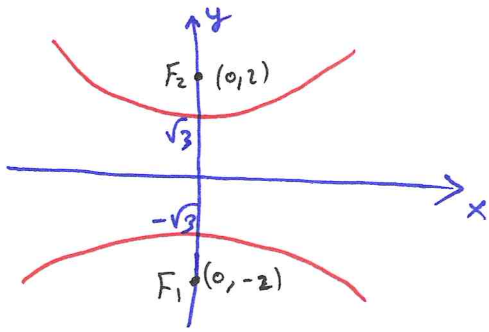
   $c = \sqrt{a^2+b^2} = \sqrt{2+8} = \sqrt{10}$

   Asymptotes $y = \pm 2x$ $\dfrac{b}{a} = 2$



2. $y^2 - 3x^2 = 3 \Leftrightarrow \dfrac{y^2}{3} - x^2 = 1$

   $c = \sqrt{3+1} = 2$, Asymptotes $y = \pm\sqrt{3}x$



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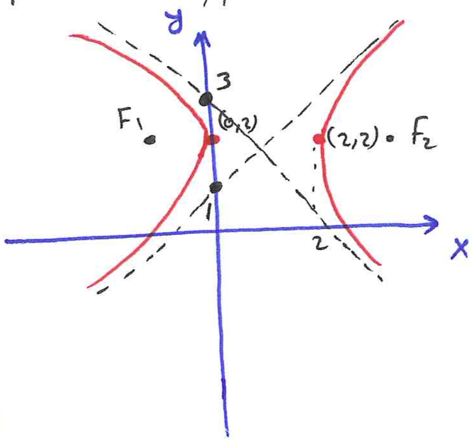
<u>Exp</u> show that $x^2 - y^2 - 2x + 4y = 4$ represents a hyperbola. Find it's center, asymptotes and foci.



$$(x-1)^2 - 1 - (y-2)^2 + 4 = 4$$

$$(x-1)^2 - (y-2)^2 = 1$$

- center $(1,2)$
- $c = \sqrt{a^2+b^2} = \sqrt{2}$ $\Rightarrow$ foci: $(1 \pm \sqrt{2}, 2)$
- Asymptotes $y - 2 = \pm (x-1)$
- Vertices: $(0,2)$ and $(2,2)$ $: (1 \pm \underset{1}{\underset{\downarrow}{a}}, 2)$

$$\frac{(y-3)^2}{3} - (x-1)^2 = 1$$

- Center $(1,3)$
- Vertices $(1, 3 \pm a) = (1, 3 \pm \sqrt{3})$
- foci : $(1, 3 \pm c)$, $c = \sqrt{a^2+b^2}$
  $(1,1), (1,5)$ $= \sqrt{3+1} = 2$
- Asymptotes $y - 3 = \pm \sqrt{3}(x-1)$
