0.4 Radicals and Rational Exponents
الجذور النسبية

if we have $b^3 = 8$

$b = \sqrt[3]{8} = \sqrt[3]{2.2.2} = 2$

$b = 2$

In general, the n^{th} root of a is $\sqrt[n]{a}$

$\sqrt[n]{a} = b$ only if $b^n = a$

n: index , a: radicand

Examples: ① $\sqrt{4} = \sqrt[2]{4} = \sqrt[2]{2.2} = 2$

② $\sqrt[3]{64} = \sqrt[3]{4.4.4} = 4$

③ $\sqrt[3]{-27} = \sqrt[3]{-3.-3.-3} = -3$

④ $\sqrt[2]{-9} = \sqrt[2]{-3.3}$ not real

If n is even and $a < 0$, then $\sqrt[n]{a}$ is not real

Definition: For a positive integer n

$$\sqrt[n]{a} = a^{\frac{1}{n}}$$

Example: $9^{\frac{1}{2}} = \sqrt[2]{9} = \sqrt[2]{3.3} = 3$

$(16)^{\frac{1}{4}} = \sqrt[4]{16} = \sqrt[4]{2.2.2.2} = 2$

$(-25)^{\frac{1}{2}} = \sqrt[2]{-25}$ not real.

Definition. For a positive integer n

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{a}\right)^m$$

or

$$a^{\frac{m}{n}} = \left(a^m\right)^{\frac{1}{n}} = \sqrt[n]{a^m}$$

Example: Write the following in radical form and simplify اكتب التالي على شكل جذر ثم بسط المقدار

① $16^{\frac{3}{4}} = \left(16^{1/4}\right)^3 = \left(\sqrt[4]{16}\right)^3 = (2)^3 = 8$

② $y^{-3/2} = \dfrac{1}{y^{3/2}} = \dfrac{1}{(y^3)^{1/2}} = \dfrac{1}{\sqrt[2]{y^3}}$

③ $\cancel{\sqrt[3]{(ab)^3}} \neq \cancel{\left((ab)^3\right)^{1/3}} = \cancel{(ab)^{3/3}} = \cancel{ab}$

$(6m)^{2/3} = \sqrt[3]{(6m)^2} = \sqrt[3]{(6)^2 m^2} = \sqrt[3]{36\, m^2}$

Example: Write the following without radical signs:

① $\sqrt{x^3} = \left(x^3\right)^{1/2} = x^{3/2}$

② $\dfrac{1}{\sqrt[3]{b^2}} = \dfrac{1}{(b^2)^{1/3}} = \dfrac{1}{b^{2/3}} = b^{-2/3}$

③ $\sqrt[3]{(ab)^3} = (ab)^{3/3} = ab$

Rules for radicals: if $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real

① $\sqrt[n]{a^n} = a$

This does not apply to $\sqrt[2]{(-2)^2}$ because $\sqrt{-2}$ not real

so $\sqrt{a^2} = |a|$

② $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

Ex: $\sqrt[3]{2} \cdot \sqrt[3]{4} = \sqrt[3]{2 \cdot 4} = \sqrt[3]{8} = 2$

③ $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$, $b \neq 0$

Ex: $\dfrac{\sqrt[3]{54}}{\sqrt[3]{2}} = \sqrt[3]{\dfrac{54}{2}} = \sqrt[3]{27} = \sqrt[3]{3 \cdot 3 \cdot 3} = 3$

Simplifying Radicals:

A radical is considered simplified if its radicand has no factors with powers $\geqslant$ index.

Ex: ① $\sqrt[3]{x^3} = x^{3/3} = x$

② $\sqrt[2]{49x} = \sqrt[2]{(7)^2 x} = \sqrt[2]{7 \cdot 7 \cdot x} = 7\sqrt[2]{x}$

③ $\sqrt[2]{6x^4} = \sqrt[2]{6x^2 \cdot x^2} = \sqrt[2]{6}\, x^2$

or $\sqrt[2]{6} \cdot \sqrt[2]{x^4} = \sqrt[2]{6} \cdot x^{4/2} = \sqrt[2]{6} \cdot x^2$

Question. Find the following and simplify the answer

① $\sqrt{8xy^3z} \cdot \sqrt{4x^2y^3z^2}$

$$= \sqrt{8 \cdot 4 \cdot x \cdot x^2 \cdot y^3 \cdot y^3 \cdot z \cdot z^2}$$

$$= \sqrt{16 \cdot 2 \cdot x \cdot x^2 \cdot y^6 \cdot z \cdot z^2}$$

$$= \sqrt{16x^2y^6z^2} \cdot \sqrt{2xz} \quad = 4xy^3z\sqrt{2xz}$$

② $\dfrac{\sqrt[3]{-16x^3y^4}}{\sqrt[3]{128y^2}} = \sqrt[3]{\dfrac{-16x^3y^4}{128y^2}} = \sqrt[3]{\dfrac{-\cancel{16}x^3y^4}{8 \cdot \cancel{16} \cdot y^2}}$

$$= \sqrt[3]{\frac{-x^3y^4}{8y^2}} = \sqrt[3]{\frac{-x^3y^{4-2}}{8}} = \sqrt[3]{\frac{-x^3y^2}{8}}$$

$$= \frac{\sqrt[3]{-x^3}\,\sqrt[3]{y^2}}{\sqrt[3]{8}} = \frac{-x\sqrt[3]{y^2}}{2}$$