1.5 Solutions of Systems of Linear Equations

A solution of a linear equation with two variables x and y is the point (x, y) that will make the equation true.

Example: For the equation $4x + 3y = 11$

Is $(0, \frac{11}{3})$ a solution?

$$4(0) + 3(\tfrac{11}{3}) \overset{??}{=} 11$$
$$0 + 11 \overset{??}{=} 11$$
$$11 = 11 \checkmark$$ it is a solution

Is $(\frac{11}{4}, 0)$ a solution?

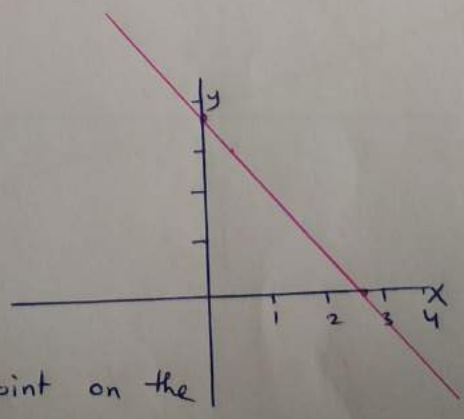
$$4(\tfrac{11}{4}) + 3(0) \overset{??}{=} 11$$
$$11 + 0 \overset{??}{=} 11$$
$$11 = 11 \checkmark$$ it is a solution

Is $(\frac{2}{4}, 3)$ a solution?

$$4(\tfrac{2}{4}) + 3(3) \overset{??}{=} 11$$
$$2 + 9 \overset{??}{=} 11$$
$$11 = 11 \checkmark$$

Graph the equation

x	y
0	$\frac{11}{3}$
$\frac{11}{4}$	0



In fact any point on the graph of the line is a solution.

System of equations (more than one equation):

A solution that satisfies more than equation is called a simultaneous solution and it could be found using three methods:

1) Graphically.
2) By substitution
3) By Elimination: the coefficient of the variables to be eliminated should be the same in both equations.

Example: Find a simultaneous solution for the following system of equations.
$$\begin{cases} 4x + 3y = 11 \\ 2x - 5y = -1 \end{cases}$$

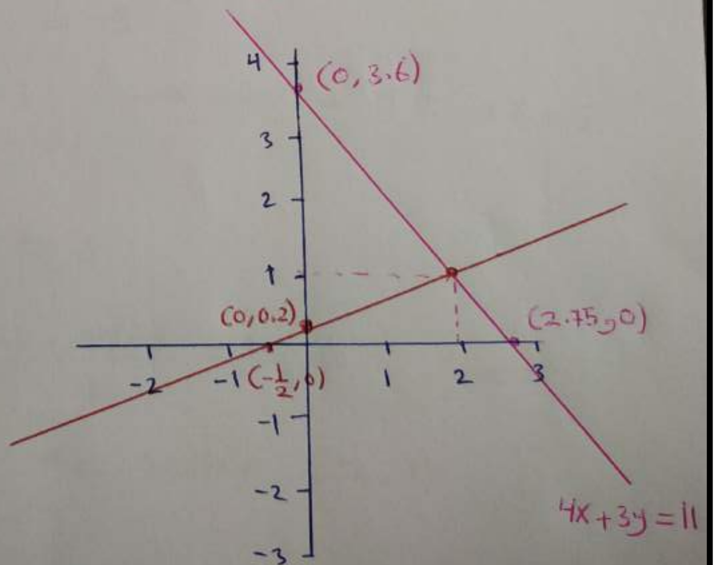
1) Graphically:

$4x + 3y = 11$

x	y
0	$\frac{11}{3} \simeq 3.6$
$2.75 \simeq \frac{11}{4}$	0

$2x - 5y = -1$

x	y
0	$\frac{1}{5} \simeq 0.2$
$-\frac{1}{2}$	0



The solution (2, 1)

2) By Substitution: $\begin{cases} 4x + 3y = 11 & ----① \\ 2x - 5y = -1 & ----② \end{cases}$

In eq. ① $4x + 3y = 11 \rightarrow 3y = 11 - 4x$

$\rightarrow y = \dfrac{11 - 4x}{3}$ --- ③

Substitute eq. ③ in eq ②

$2x - 5y = -1$

$2x - 5\left(\dfrac{11-4x}{3}\right) = -1$ multiply by 3

$3.(2x) - 3\left(5\left(\dfrac{11-4x}{3}\right)\right) = 3.(-1)$

$6x - 5(11 - 4x) = -3$

$6x - 55 + 20x = -3$

$$\begin{array}{l} 26x - 55 = -3 \\ \quad\;\; +55 \quad\;\; +55 \end{array} \rightarrow \dfrac{26x}{26} = \dfrac{52}{26} \rightarrow x = 2$$

Substitute $x=2$ in eq. 3

$y = \dfrac{11 - 4x}{3} = \dfrac{11 - 4(2)}{3} = \dfrac{11-8}{3} = \dfrac{3}{3} = 1$

The solution of the system $(2, 1)$

3) By Elimination: $\begin{cases} 4x + 3y = 11 & \text{--- } ① \\ 2x - 5y = -1 & \text{--- } ② \end{cases}$

Multiply eq. ② by -2, you get $-4x + 10y = 2$

- You have the system

$$\begin{array}{r} 4x + 3y = 11 \\ -4x + 10y = 2 \\ \hline \end{array}$$

- add the two equations: $\frac{13y}{13} = \frac{13}{13} \rightarrow \boxed{y = 1}$

- Substitute $y = 1$ in eq ① $4x + 3y = 11$

$4x + 3(1) = 11 \rightarrow 4x + 3 = 11$
$ -3 \quad -3$

$\rightarrow \frac{4x}{4} = \frac{8}{4} \rightarrow \boxed{x = 2} \rightarrow$ The solution $(2, 1)$

If you want to check the solution substitute $(2,1)$ in the second equation $2x - 5y = -1$

$2(2) - 5(1) \overset{??}{=} -1$

$4 - 5 \overset{??}{=} -1$

$-1 = -1$ ✓

Example: Solve the following system of linear equations

$$\begin{cases} 3x + 4y = 1 & \text{---- ①} \\ 2x - 3y = 12 & \text{---- ②} \end{cases}$$

By elimination: multiply eq ① by 3 and multiply eq. ② by 4

You have
$$9x + 12y = 3$$
$$8x - 12y = 48$$

add the two equations: $\dfrac{17x}{17} = \dfrac{51}{17} \rightarrow \boxed{x = 3}$

Substitute $x = 3$ in eq ②: $8x - 12y = \cancel{12}48$

$8(3) - 12y = 48 \rightarrow 24 - 12y = 48$
$\qquad\qquad\qquad\qquad -24 \qquad\quad -24$

$\rightarrow \dfrac{-12y}{-12} = \dfrac{24}{-12} \rightarrow \boxed{y = -2}$

The solution is $(3, -2)$

check the solution: substitute $(3, -2)$ in eq ①

$9x + 12y = 3 \rightarrow 9(3) + 12(-2) \overset{??}{=} 3$

$27 - 24 \overset{??}{=} 3$

$3 = 3$ ✓

Example: Solve the following $\begin{cases} \cancel{8}4x + 3y = 4 & \text{---}(1) \\ 8x + 6y = 18 & \text{---}(2) \end{cases}$

This system is called inconsistent system (Note that they have the same slope)

$4x + 3y = 4 \rightarrow y = \frac{4}{3} - \frac{4x}{3} \rightarrow$ Slope $= -\frac{4}{3}$

$8x + 6y = 18 \rightarrow y = \frac{18}{6} - \frac{8x}{6} \rightarrow$ Slope $= -\frac{8}{6} = -\frac{4}{3}$

Multiply eq (1) by -2, You get $-8x - 6y = -8$

You have
$$-8x - 6y = -8$$
$$8x + 6y = 18$$

add the two equations: $0 + 0 = 10$

$$\boxed{0 = 10}$$

There are no solutions of the system, their graphs are parallel lines.

Example: Solve the following $\begin{cases} 4x + 3y = 4 & \text{--- ①} \\ 8x + 6y = 8 & \text{--- ②} \end{cases}$

This system is called independent system. Note that they have the same slope and multiple of each other (eq② = 2(eq①))

Multiply eq ① by -2, you get $-8x - 6y = -8$

You have
$$-8x - 6y = -8$$
$$8x + 6y = 8$$

add the equations : $0 + 0 = 0 \rightarrow \boxed{0 = 0}$

So the system has infinitely many solutions, their graphs coincide and each point on this graph represents a solution of the system.

Question: Try to solve the following system:

$$\begin{cases} 2x - 4 = 5y & \text{--- ①} \\ x + 2y = 3 & \text{--- ②} \end{cases}$$

You can rearrang eq①, $2x - 5y = 4$