4.2 The Optimal Values of a Linear Function Subject to Constraints

For a closed and bounded region, we can find the optimal (maximum or minimum) value of the objective function by evaluating the function at each of the corners of the solution region.

Example:- Find the maximum and minimum value of $C = 2x + y$ subject to the constraints

$$\begin{cases} x + 2y \le 10 \\ 2x + y \le 14 \\ x \geq 0, \quad y \geq 0 \end{cases}$$

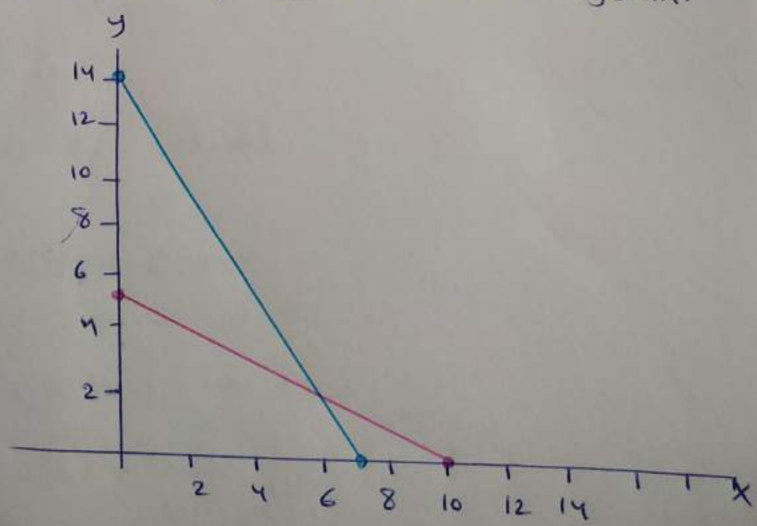
① Graph the solution of the constraint system:-

$x + 2y = 10$

x	y
0	5
10	0

$2x + y = 14$

x	y
0	14
7	0

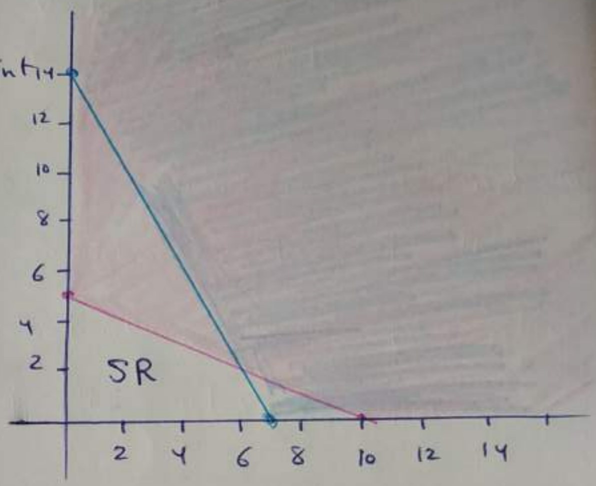


$x \geq 0, \; y \geq 0 \rightarrow$ Quadrant I for $x + 2y \le 10$

take $(0,0)$ as a test point

$0 + 2(0) \overset{??}{\le} 10$

$0 \le 10$ ✓



for $2x + y \le 14$

take $(0,0)$ as a test point

$2(0) + 0 \overset{??}{\le} 14$

$0 \le 14$ ✓

② Corners : $(0,5)$, $(0,0)$, $(7,0)$

$2x + y = 14$
$(x + 2y = 10) \times -2$ → $2x + y = 14$
$-2x - 4y = -20$ → $-3y = -6$

→ $\boxed{y = 2}$

$2x + y = 14 \rightarrow 2x + 2 = 14 \rightarrow 2x = 12$

→ $\boxed{x = 6}$

The fourth corner is $(6,2)$

③ At $(0,0)$: $C = 2(0) + 0 = 0$

At $(0,5)$: $C = 2(0) + 5 = 5$

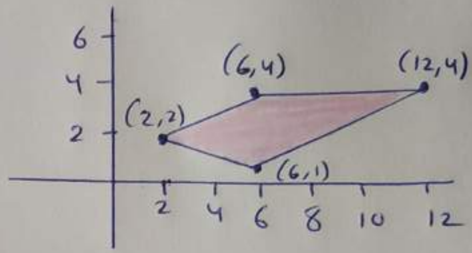
At $(7,0)$: $C = 2(7) + 0 = 14$

At $(6,2)$: $C = 2(6) + 2 = 18$

$C = 18$ is the maximum value at $x = 6$, $y = 2$

$C = 0$ is the minimum value at $x = 0$, $y = 0$

Example:- Use the given feasible region determined by the constraint inequalities to find the maximum and minimum of the objective function $f = 2x + 7y$ (if they exist)



At $(2,2)$:- $f = 2(2) + 7(2)$

$f = 4 + 14 = 18$

At $(6,4)$:- $f = 2(6) + 7(4)$

$f = 12 + 28 = 40$

At $(6,1)$:- $f = 2(6) + 7(1) = 12 + 7 = 19$

At $(12,4)$:- $f = 2(12) + 7(4) = 24 + 28 = 52$

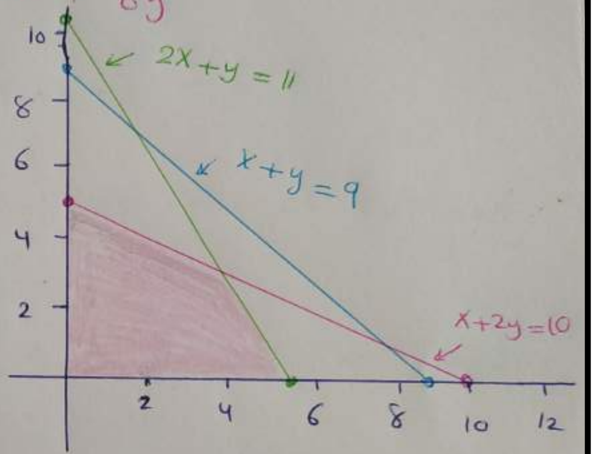
f has a maximum value at $x = 12$ and $y = 4$

f has a minimum value at $x = 2$ and $y = 2$

The maximum value of f is 52

The minimum value of f is 18

Example :- The graph of the feasible region is shown. Find the corners of ~~each~~ the feasible region, and then find the maximum and minimum of the objective function $g = 12x + 8y$



Corners :

at $x + 2y = 10$

y-intercept : put $x = 0$

$0 + 2y = 10 \rightarrow (y = 5)$

The first corner is $(0, 5)$

on $2x + y = 11$

x-intercept : put $y = 0$

$2x + 0 = 11 \rightarrow x = \frac{11}{2} = 5.5$

The second corner is $(5.5, 0)$

The third corner on $x + 2y = 10$ and $2x + y = 11$

$$\begin{aligned} x + 2y &= 10 \\ \times -2\,(2x + y &= 11) \end{aligned} \rightarrow \begin{aligned} x + 2y &= 10 \\ -4x - 2y &= -22 \end{aligned} \rightarrow -3x = -12 \rightarrow x = 4$$

$x + 2y = 10 \rightarrow 4 + 2y = 10 \rightarrow y = \frac{6}{2} = 3$

The third corner is $(4, 3)$

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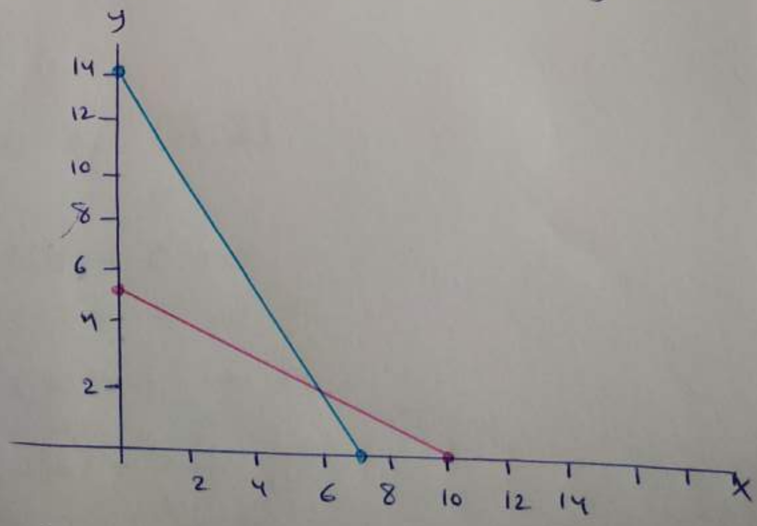

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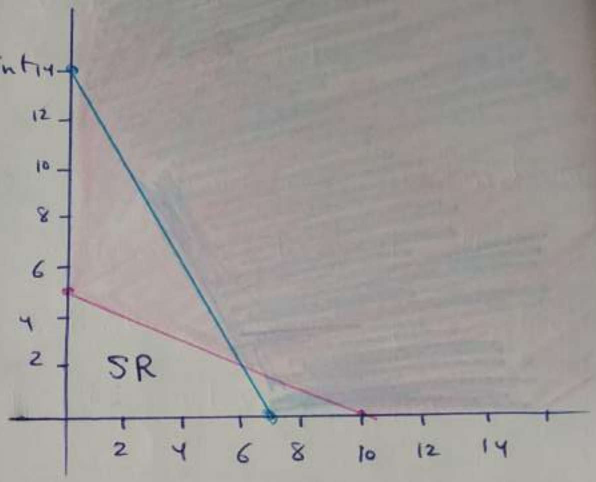

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$\rightarrow \boxed{y = 2}$

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$\rightarrow \boxed{x = 6}$

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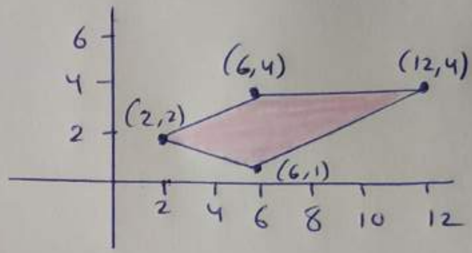

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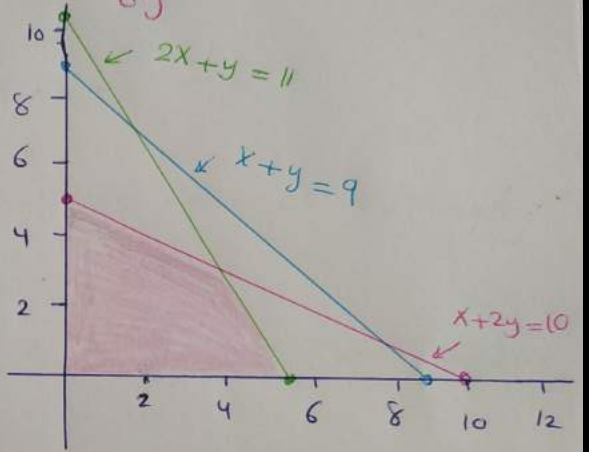

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The third corner on $x + 2y = 10$ and $2x + y = 11$

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$$x + 2y = 10 \rightarrow 4 + 2y = 10 \rightarrow y = \frac{6}{2} = 3$$

The third corner is $(4, 3)$

At $(0,5)$:-

$g = 12(0) + 8(5) = 40$

At $(5.5, 0)$:-

$g = 12(5.5) + 8(0) = 66$

At $(4, 3)$:-

$g = 12(4) + 8(3) = 48 + 24 = 72$

The maximum value of g is 72 at $x = 4$ and $y = 3$

The minimum value of g is 40 at $x = 0$ and $y = 5$