Math141-Calculus I: Review of differentiation and integration
Lecture notes based on Thomas Calculus Book
Chapter 1 to Chapter 5

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September 16, 2017

Chapter 1

Functions

[1]

1.1 Functions

In this lecture, we review some important functions with their domains, ranges and graphs.

Definition 1.1.1 *A function f is a rule that assigns to each point x in the domain a unique point $y = f(x)$ in the range of f. We write $f : D \to R$ where D is the domain of f and R is its range.*

Example 1.1.1 (a) $f(x) = x^2$, $D = (-\infty, \infty)$, $R = [0, \infty)$.

(b) $f(x) = \sqrt{x}$, $D = R = [0, \infty)$.

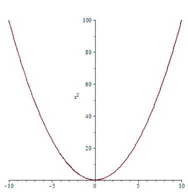

Figure 1.1: Graph of $y = x^2$

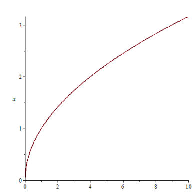

Figure 1.2: Graph of $y = \sqrt{x}$

(c) $f(x) = \sqrt{1 - x^2}$, $D = [-1, 1], R = [0, 1]$.

(d) The absolute value function $f(x) = |x| = \sqrt{x^2}$, $D = (-\infty, \infty)$, $R = [0, \infty)$.

(e) The greatest integer function $f(x) = \lfloor x \rfloor$, $D = (-\infty, \infty)$, $R = 0, \pm 1, \pm 2, ...$

[1] review of chapter 1 in the textbook

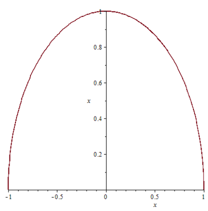

Figure 1.3: Graph of $y = \sqrt{1 - x^2}$

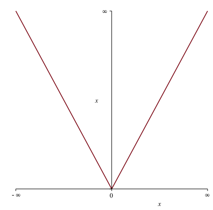

Figure 1.4: Graph of $y = |x|$

1.2 Trigonometric functions

In this section, we review the six trigonometric functions: $\sin x, \cos x, \tan x, \cot x, \sec x$ and $\csc x$. You are supposed to know the values of these functions at the main values $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, ...$

(a) $y = \sin x,\ D = (-\infty, \infty),\ R = [-1, 1]$.

(b) $y = \cos x,\ D = (-\infty, \infty),\ R = [-1, 1]$.

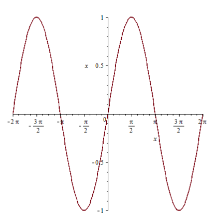

Figure 1.5: Graph of $y = \sin x$

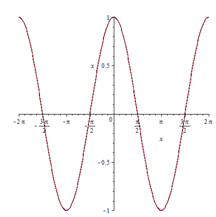

Figure 1.6: Graph of $y = \cos x$

(c) $y = \tan x = \frac{\sin x}{\cos x},\ D = (-\infty, \infty) \setminus \{\frac{\pi}{2} \pm n\pi\},\ n = 0, 1, 2, ...,\ R = (-\infty, \infty)$

(d) $y = \cot x = \frac{\cos x}{\sin x},\ D = (-\infty, \infty) \setminus \{\pm n\pi\},\ n = 0, 1, 2, ...,\ R = (-\infty, \infty)$

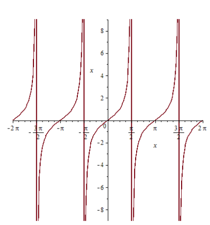

Figure 1.7: Graph of $y = \tan x$

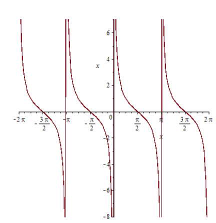

Figure 1.8: Graph of $y = \cot x$

(e) $y = \sec x = \frac{1}{\cos x},\ D = (-\infty, \infty) \setminus \{\frac{\pi}{2} \pm n\pi\},\ n = 0, 1, 2, ...,$
$R = (-\infty, -1] \cup [1, \infty)$

(f) $y = \csc x = \frac{1}{\sin x}$, $D = (-\infty, \infty) \setminus \{\pm n\pi\}$, $n = 0, 1, 2, ...$,
$R = (-\infty, -1] \cup [1, \infty)$

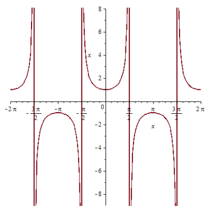

Figure 1.9: Graph of $y = \sec x$

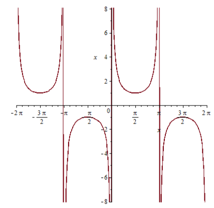

Figure 1.10: Graph of $y = \csc x$

Remark 1.2.1 Since $\sin(x+2\pi) = \sin x$, $\cos(x+2\pi) = \cos x$, $\sec(x+2\pi) = \sec x$ and $\csc(x + 2\pi) = \csc x$, the functions $\sin x, \cos x, \sec x$ and $\csc x$ are called periodic with period 2π. Whereas $\tan x$ and $\cot x$ are periodic with period π since $\tan(x + \pi) = \tan x$ and $\cot(x + \pi) = \cot x$.

1.2.1 Trigonometric identities

1. $\sin^2 x + \cos^2 x = 1$.
2. $\sin(2x) = 2 \sin x \cos x$.
3. $\cos(2x) = \cos^2 x - \sin^2 x$.
4. $\cos^2 x = \frac{1+\cos(2x)}{2}$.
5. $\sin^2 x = \frac{1-\cos(2x)}{2}$.
6. $\sec^2 x = 1 + \tan^2 x$.
7. $\csc^2 x = 1 + \cot^2 x$.
8. $\cos(A + B) = \cos A \cos B - \sin A \sin B$.
9. $\sin(A + B) = \sin A \cos B + \cos A \sin B$.

Example 1.2.1 Using the above identities, we find the following:

(a) $\sin(x + \pi) = -\sin x$, $\cos(x + \pi) = -\cos x$.

(b) $\sin(x + \frac{\pi}{2}) = \cos x$, $\cos(x + \frac{\pi}{2}) = -\sin x$.

1.3 Even and odd functions

Definition 1.3.1 *Let f be a function defined on an interval $I = [-a, a]$, where a is a positive real number. Then*

- *$f(x)$ is called even if $f(-x) = f(x)$. If f is even then its graph is symmetric about the $y-$axis.*
- *$f(x)$ is called odd if $f(-x) = -f(x)$. If f is odd then its graph is symmetric about the origin.*

Example 1.3.1 $x^2, x^4, x^6, ..., \cos x, \sec x$ are even. $x, x^3, x^5, ..., \sin x, \tan x, \csc x, \cot x$ are odd.

1.3.1 Exercises

(1) Find the domain and the range of the following functions:

(a) $f(x) = \frac{1}{\sqrt{x}}$.
(b) $f(x) = \tan(\pi x)$.
(c) $f(x) = 1 + |x|$.
(d) $f(x) = \sec^2 x$.
(e) $g(x) = \frac{1}{x^2}$.
(f) $h(x) = \frac{1}{\sqrt{1-x^2}}$.

(2) Sketch the following functions:

(a) $y = \sin(\pi x)$
(b) $y = |x - 1|$
(c) $y = \cos(x) + 1$

(3) Determine whether the following functions are even, odd or neither:

(a) $f(x) = x^2 + 1$.
(b) $f(x) = x^3 + x$.
(c) $g(t) = \frac{1}{t-1}$.
(d) $h(x) = \frac{x}{x^2-1}$.

(4) Prove the following:

(a) If $f(x)$ is even and $g(x)$ is odd then $(g \circ f)(x)$ is even.
(b) If $f(x)$ is even and $g(x)$ is odd then $\frac{f(x)}{g(x)}$ is odd.

Chapter 2

Limits and continuity

[1]

2.1 Limits of functions

When a function f approaches a certain limit L as x approaches x_0, we write

$$\lim_{x \to x_0} f(x) = L$$

This limit means that *the function gets arbitrarily close to* L *when* x *is sufficiently close to* x_0. Notice that x_0 or L or both of them can be $+\infty$ or $-\infty$. The function f may or may not be defined at x_0. As you know,

$$\lim_{x \to x_0} f(x) = L \quad \text{if and only if} \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = L$$

Example 2.1.1 We can use simple techniques to find the following limits:

(a) $\lim_{x \to 1} \frac{x-1}{x+1} = 0$.

(b) $\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$.

(c) $\lim_{x \to +\infty} \frac{1}{x} = 0$.

(d) $\lim_{x \to 0^+} \frac{1}{x} = +\infty$.

(e) $\lim_{x \to 1} \frac{x^2+x-2}{x^2-x} = 3$.

(f) $\lim_{x \to -1} \frac{\sqrt{x^2+8}-3}{x+1} = -\frac{1}{3}$.

[1] This is a review of chapter two in the textbook

Theorem 2.1.1 ***(The Sandwich Theorem)*** *Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ and that*

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L \quad \textit{then} \quad \lim_{x \to c} f(x) = L$$

Example 2.1.2 Suppose that $f(x)$ is a function that satisfies $1 - x^2 \leq f(x) \leq 1 + x^2$. Then $\lim\limits_{x \to 0} f(x) = 1$ since $\lim\limits_{x \to 0}(1 - x^2) = \lim\limits_{x \to 0}(1 + x^2) = 1$.

Example 2.1.3 Find $\lim\limits_{x \to +\infty} \frac{\sin x}{x}$. Since

$$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

and $\lim\limits_{x \to \infty} \frac{1}{x} = 0$, then, by the sandwich theorem

$$\lim_{x \to \infty} \frac{\sin x}{x} = 0$$

Remark 2.1.1 Please do not confound the previous limit with $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$.

Example 2.1.4 Consider the function

$$f(x) = \begin{cases} x + 1 & , \quad x \leq 0 \\ -x & , \quad x > 0 \end{cases}$$

Then, $\lim_{x \to 0^+} f(x) = 0$ and $\lim_{x \to 0^-} f(x) = 1$. So, $\lim_{x \to 0} f(x)$ does not exist.

2.2 Continuity

Definition 2.2.1 *A function f is continuous at a point x_0 if the following conditions are satisfied:*

(a) $f(x_0)$ *exists.*

(b) $\lim\limits_{x \to x_0} f(x)$ *exists.*

(c) $\lim\limits_{x \to x_0} f(x) = f(x_0)$.

Example 2.2.1 The functions $\sin x, \cos x$, $|x|$ and all polynomials are continuous on $(-\infty, \infty)$.

Example 2.2.2 The rational functions are continuous at all points except at the zeros of the denominator. For example, the function

$$f(x) = \frac{x^3 + x + 1}{x^2 - 1}$$

is continuous on $(-\infty, \infty) \setminus \{-1, 1\}$.

Example 2.2.3 (a function with removable discontinuity) Consider the function

$$f(x) = \frac{x^2 + 2x - 3}{x^2 - 1}$$

Then

$$\lim_{x \to 1} \frac{x^2 + 2x - 3}{x^2 - 1} = \lim_{x \to 1} \frac{(x-1)(x+3)}{(x-1)(x+1)} = \lim_{x \to 1} \frac{x+3}{x+1} = 2$$

The point $x = 1$ is called a **removable discontinuity** of the function f because we can define f at $x = 1$ so that we can remove the discontinuity. The following function is called the **continuous extension of f at** $x = 1$

$$F(x) = \begin{cases} f(x) & , \quad x \neq 1 \\ 2 & , \quad x = 1 \end{cases}$$

Theorem 2.2.1 ***(The intermediate value theorem)*** *If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.*

Recall that a point c is called a root of a function f if $f(c) = 0$. We can use the intermediate value theorem to show that a given function has a root in some interval.

Example 2.2.4 Let $f(x) = x^3 - x - 1$. Since $f(1) = -1 < 0$, $f(2) = 5 > 0$ and $f(1) < 0 < f(2)$ then there exists $c \in [1, 2]$ such that $f(c) = 0$.

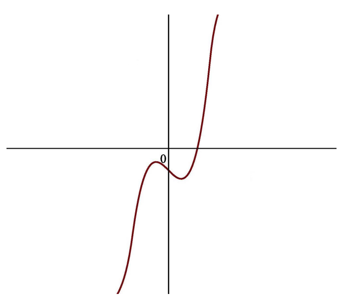

Figure 2.1: Graph of $y = x^3 - x - 1$

2.2.1 Asymptotes

In this section, we are dealing mainly with rational functions. A rational function is the ratio of two polynomials. Our objective is to be able to sketch some rational functions using limits and asymptotes.

Definition 2.2.2 *A line $y = b$ is a horizontal asymptote of the graph of the function $y = f(x)$ if either*

$$\lim_{x\to\infty} f(x) = b \quad or \quad \lim_{x\to-\infty} f(x) = b$$

Example 2.2.5 The line $y = 0$ is a horizontal asymptote for $f(x) = \frac{x}{x^2+1}$ since $\lim\limits_{x\to+\infty} \frac{x}{x^2+1} = \lim\limits_{x\to-\infty} \frac{x}{x^2+1} = 0$.

Example 2.2.6 The line $y = 1$ is a horizontal asymptote for $f(x) = \frac{x^2}{x^2+1}$ since $\lim\limits_{x\to+\infty} \frac{x^2}{x^2+1} = \lim\limits_{x\to-\infty} \frac{x^2}{x^2+1} = 1$.

Definition 2.2.3 *A line $x = a$ is a vertical asymptote of the graph of the function $y = f(x)$ if either*

$$\lim_{x\to a^+} f(x) = \pm\infty \quad or \quad \lim_{x\to a^-} f(x) = \pm\infty$$

Example 2.2.7 The line $x = 0$ is a vertical asymptote for $f(x) = \frac{1}{x}$ since $\lim\limits_{x\to0^+} \frac{1}{x} = +\infty$ and $\lim\limits_{x\to0^-} \frac{1}{x} = -\infty$.

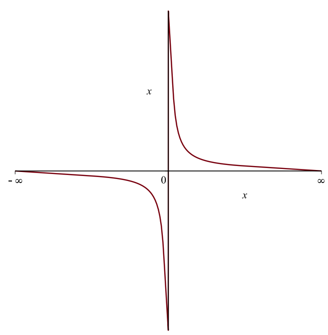

Figure 2.2: Graph of $y = \frac{1}{x}$

Example 2.2.8 The function $f(x) = \frac{\sin x}{x}$ has no vertical asymptote even it is undefined at $x = 0$ since $\lim\limits_{x\to0} \frac{\sin x}{x} = 1$.

Example 2.2.9 Consider the function $f(x) = \frac{x+1}{x-1}$. Notice that

$$\lim_{x\to1^+} \frac{x+1}{x-1} = +\infty, \quad \lim_{x\to1^-} \frac{x+1}{x-1} = -\infty$$

and

$$\lim_{x\to+\infty} \frac{x+1}{x-1} = \lim_{x\to-\infty} \frac{x+1}{x-1} = 1$$

Then the line $x = 1$ is a vertical asymptote and the line $y = 1$ is a horizontal asymptote.

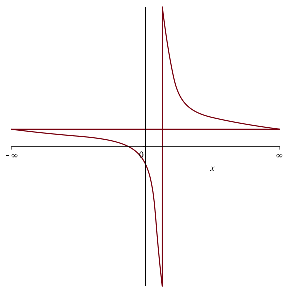

Figure 2.3: Graph of $y = \frac{x+1}{x-1}$

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator then the graph of f has an **oblique asymptote**.

Example 2.2.10 The graph of the function $f(x) = \frac{x^2}{x-1}$ has an oblique asymptote since the degree of the numerator is 2 and the degree of the denominator is one. Using polynomial division, we can write $f(x) = (x+1) + \frac{1}{x-1}$ So, the line $y = x + 1$ is the oblique asymptote of the graph of f. Moreover, the line $x = 1$ is a vertical asymptote for the graph of f since $\lim_{x \to 1^+} f(x) = +\infty$ and $\lim_{x \to 1^-} f(x) = -\infty$.

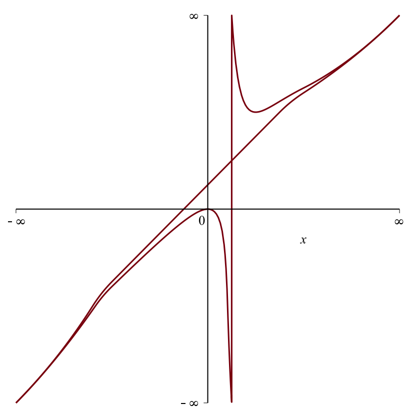

Figure 2.4: Graph of $y = \frac{x^2}{x-1}$

2.3 Exercises

1. Find the following limits:

   (a) $\lim_{t\to -1} \frac{t^2+3t+2}{t^2-t-2}$

   (b) $\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}$

   (c) $\lim_{\theta\to 1} \frac{\theta^4-1}{\theta^3-1}$

   (d) $\lim_{\theta\to 0} \frac{\sin(2\theta)}{3\theta}$

   (e) $\lim_{\theta\to 0} \frac{1-\cos\theta}{\sin(2\theta)}$

   (f) $\lim_{x\to\infty} \frac{1+\sqrt{x}}{1-\sqrt{x}}$

   (g) $\lim_{x\to -\infty} \frac{\sqrt{x^2+1}}{x+1}$

   (h) $\lim_{x\to -\infty} \frac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}}$

   (i) $\lim_{x\to\infty} (\sqrt{x^2+1}-\sqrt{x^2-x})$

   (j) $\lim_{t\to 3^+} \frac{\lfloor t\rfloor}{t}$

   (k) $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$

2. Find the asymptotes of the following functions then sketch their graphs

   (a) $f(x) = \frac{x+1}{x-1}$

   (b) $y = \frac{x^3+1}{x^2}$

   (c) $f(x) = \frac{x^2+1}{x-1}$

   (d) $f(x) = \frac{x^3+1}{x^2-1}$

3. For what values of a and b is

$$g(x) = \begin{cases} ax+2b & , \quad x \le 0 \\ x^2+3a-b & , \quad 0 < x \le 2 \\ 3x-5 & , \quad x > 2 \end{cases}$$

   continuous at every x. Then sketch the graph of the function.

4. Find the continuous extension of the function $h(t) = \frac{t^2+3t-10}{t-2}$.

5. Use the intermediate value theorem to show that the function $f(x) = x^3-2x^2+2$ has a root.

Chapter 3

Differentiation

[1]

3.1 Definition of derivative

Definition 3.1.1 *The derivative of a function f at x_0, denoted $f'(x_0)$ is*

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists.

If $f'(x_0)$ exists then we say that f is **differentiable** at x_0. When we say that f is differentiable on a closed interval $[a, b]$, we mean the following

- f' exists at all points in the open interval (a, b).
- The **right-hand derivative of f at** a exists; that is,

$$\lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h}$$

  exists. We denote the right-hand derivative of f at $x = a$ by $f'_+(a)$.
- The **left-hand derivative of f at** b exists; that is,

$$\lim_{h \to 0^-} \frac{f(a + h) - f(a)}{h}$$

  exists. We denote the left-hand derivative of f at $x = b$ by $f'_-(b)$.

Remark 3.1.1 A function f is differentiable at $x = c$ if and only if the right-hand derivative and the left-hand derivative both exist and are equal at $x = c$.

[1]This is a review of chapter 3 in the textbook

If f is differentiable at $x = c$ then f is continuous at $x = c$. The converse of this statement is not true, the function $f(x) = |x|$ is continuous but not differentiable at $x = 0$.

Example 3.1.1 Let $f(x) = |x|$. We find the left-hand and right-hand derivatives of f at $x = 0$.

$$f'_+(0) = \lim_{h\to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$$

$$f'_-(0) = \lim_{h\to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$$

We conclude that f is not differentiable at $x = 0$.

3.2 Differentiation rules

Theorem 3.2.1 *Suppose that $f(x)$ and $g(x)$ are differentiable at x. Then*

1. $(f(x) \pm g(x))' = f'(x) \pm g'(x)$.
2. $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.
3. $\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$.
4. $(f \circ g)'(x) = f'(g(x))g'(x)$ *(Chain Rule).*

3.3 Derivatives of Trigonometric functions

1. $(\sin x)' = \cos x$.
2. $(\cos x)' = -\sin x$.
3. $(\tan x)' = \sec^2 x$.
4. $(\sec x)' = \sec x \tan x$.
5. $(\csc x)' = -\csc x \cot x$.
6. $(\cot x)' = -\csc^2 x$.

Example 3.3.1 Find the derivatives of the following functions:

1. $\frac{d}{dx}\frac{x+1}{x^2+1} = \frac{x^2+1-(x+1)(2x)}{(x^2+1)^2} = \frac{1-2x-x^2}{(x^2+1)^2}$.
2. $\frac{d}{dx}\tan(\sqrt{x}) = (\sec^2\sqrt{x})\frac{1}{2\sqrt{x}}$.
3. $\frac{d}{dx}(\sec x \tan x) = \sec^3 x + \sec x \tan^2 x$.

Example 3.3.2 Find the equation of the tangent line to the curve $f(x) = \sec x \tan x$ at $x = \frac{\pi}{4}$.

Solution: The slope of the tangent line is $f'(\frac{\pi}{4}) = 3\sqrt{2}$ (from the above example) and $f(\frac{\pi}{4}) = \sqrt{2}$.

Then, the equation of the tangent line to $f(x)$ at $x = \frac{\pi}{4}$ is

$$y - \sqrt{2} = 3\sqrt{2}(x - \frac{\pi}{4})$$

3.4 Implicit differentiation

In this section, we consider equations that define relation between x and y. We will learn how to find $\frac{dy}{dx}$ using implicit differentiation. Let us consider some examples:

Example 3.4.1 The equation $x^2 + y^2 = 1$ defines the unit circle (the circle with center $(0, 0)$ and radius one). To find y', we differentiate both sides with respect to x to get $2x + 2yy' = 0$, from which we find that $y' = -x/y$.

We can differentiate again to find the second order derivative y''.

$$y'' = \frac{d^2y}{dx^2} = \frac{-y + xy'}{y^2} = \frac{-y + x(\frac{-x}{y})}{y^2}$$

Example 3.4.2 Consider the implicit equation $xy = \cot(xy)$. Differentiate both sides with respect to x. Then

$$y + xy' = -\csc^2(xy)(y + xy')$$

From which we find that

$$\frac{dy}{dx} = \frac{-y - y\csc^2(xy)}{x + x\csc^2(xy)} = -\frac{y}{x}$$

3.5 Linearization and Differentials

Sometimes, we need to approximate a given nonlinear function with a linear function at some point near $(a, f(a))$. The best linear function that approximates $f(x)$ near $x = a$, provided that f is differentiable at $x = a$, is its tangent line whose equation is given by

$$L(x) = f(a) + f'(a)(x - a)$$

$L(x)$ is called the **linearization of** $f(x)$ **at** $x = a$ and the approximation $f(x) \approx L(x)$ is called the **standard linear approximation of** f **at** a.

Example 3.5.1 The linearization of the function $f(x) = \sqrt{1+x}$ at $x = 0$ is $L(x) = 1 + \frac{1}{2}x$. We can use the linearization to approximate the values of f near $x = 0$. For example, $\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.1$ and $\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$.

Example 3.5.2 Find the linearization of the function $f(x) = \sec x$ at $x = \frac{\pi}{4}$. We need to find $f(\frac{\pi}{4})$ and $f'(\frac{\pi}{4})$. Now, $f'(x) = \sec x \tan x$, so $f'(\frac{\pi}{4}) = \sqrt{2}$ and $f(\frac{\pi}{4}) = \sqrt{2}$. Then the linearization $L(x) = \sqrt{2} + \sqrt{2}(x - \frac{\pi}{4})$.

Now, suppose that we move from a point $x = a$ to a nearby point $a + dx$. The change in f is $\Delta f = f(a + dx) - f(a)$ while the change in L is

$$\Delta L = L(a + dx) - L(a) = f(a) + f'(a)(a + dx - a) - f(a) = f'(a)dx$$

Since $f \approx L$ then $\Delta f \approx \Delta L = f'(a)dx$. Therefore, $f'(a)dx$ gives an approximation for Δf. The quantity $f'(a)dx$ is called the **differential of f at $x = a$**. For example, the differential of the function $y = \tan^2 x$ is $dy = 2\tan x \sec^2 x dx$.

Example 3.5.3 The radius r of a circle increases from 10 to 10.1 m. Use dA to estimate the increase in the circle's area A. Estimate the area of the enlarged circle and compare your estimate to the true area found by direct calculations.
Solution The area of the circle is $A = \pi r^2$. Then $dA = 2\pi r dr$. The estimated increase is

$$dA = 2\pi(10)0.1 = 2\pi m^2$$

The estimate area of the enlarged circle is

$$A(10.1) \approx A(10) + dA = 100\pi + 2\pi = 102\pi$$

The exact value of the area is $A(10.1) = \pi(10.1)^2 = 102.01\pi$. The error in this estimation is $|102.01\pi - 102\pi| = 0.01\pi$.

3.6 Exercises

1. Find the derivatives of the following functions:
   (a) $f(s) = \frac{\sqrt{s}-1}{\sqrt{s}+1}$
   (b) $f(x) = (\frac{1}{x} - x)(x^2 + 1)$
   (c) $g(x) = \sec(2x + 1)\cot(x^2)$
   (d) $s(t) = \frac{1+\csc t}{1-\csc t}$
   (e) $f(x) = x^3 \sin x \cos x$.
   (f) $x^{1/2} + y^{1/2} = 1$.
2. Find $\frac{dy}{dx}$ for the following:

(i) $y = \cot^2 x$

(ii) $x^2 + y^2 = x$.

(iii) $y = \frac{\sin x}{1 - \cos x}$.

3. Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is parallel to the $x-$axis.

4. For what values of the constant a, if any, is

$$f(x) = \begin{cases} \sin(2x) & , \quad x \leq 0 \\ ax & , \quad x > 0 \end{cases}$$

(i) continuous at $x = 0$?

(ii) Differentiable at $x = 0$.

5. Find the normals to the curve $xy + 2x - y = 0$ that are parallel to the line $2x + y = 0$.

6. Find the linearization of the following functions at the given points

(a) $f(x) = \tan x$, $x = \pi/4$.

(b) $g(x) = \frac{1}{x}$, $x = 1$.

(c) $h(x) = \frac{x^2}{x^2+1}$, $x = 0$.

(d) $f(x) = 1 + \cos\theta$, $\theta = \frac{\pi}{3}$.

7. The radius of a circle is increased from 2 to 2.02 m.

(a) Estimate the resulting change in area.

(b) Express the estimate as a percentage of the circle's original area.

Chapter 4

Applications of derivatives

[1]

4.1 Increasing and decreasing functions

Definition 4.1.1 *Let $f(x)$ be a function defined on an interval I. Then*

(a) f is increasing on I if whenever $x_2 > x_1$ then $f(x_2) > f(x_1)$, for all x_1, x_2 in I.

(b) f is decreasing on I if whenever $x_2 > x_1$ then $f(x_2) < f(x_1)$, for all x_1, x_2 in I.

To determine whether a function f is increasing or decreasing, we use the following theorem

Theorem 4.1.1 *Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) then*

(a) If $f'(x) > 0$, for all $x \in (a, b)$ then f is increasing on $[a, b]$.

(b) If $f'(x) < 0$, for all $x \in (a, b)$, then f is decreasing on $[a, b]$.

Example 4.1.1 Let $f(x) = x^3 - 12x - 5$. Then $f'(x) = 3x^2 - 12 = 3(x-2)(x+2)$. Depending on the sign of f', we find that f is increasing on $(-\infty, -2] \cup [2, \infty)$ and decreasing on $[-2, 2]$.

[1] This part is a review of chapter 4 in the textbook

4.2 Extreme values of functions

Definition 4.2.1 *Let f be a function with domain D. Then,*

*(a) f has an **absolute maximum** value on D at a point c if $f(x) \leq f(c)$, for all $x \in D$.*

*(b) f has an **absolute minimum** value on D at a point c if $f(x) \geq f(c)$, for all $x \in D$.*

$f(c)$ is called local maximum (resp. local minimum) if the inequality in (a) (resp. (b)) holds in a small interval around $x = c$.

Example 4.2.1 The function $f(x) = x^3$, $D = [-1, 1]$ has absolute minimum value $f(-1) = -1$ and absolute maximum value $f(1) = 1$.

Theorem 4.2.1 *If f is continuous function on a closed interval $[a, b]$ then f has both an absolute maximum value and an absolute minimum value.*

If we want to find the extreme values of a function f on a closed interval, we look for these values at the endpoints of the interval and at the interior points where $f' = 0$ or undefined (**critical points**).

Definition 4.2.2 *An interior point where f' equals zero or undefined is called a critical point of f.*

Example 4.2.2 Let $f(x) = x^{2/3}$, $D = [-1, 8]$. $f'(x) = \frac{2}{3x^{1/3}}$. Then $f'(0)$ is undefined. To find the extreme values of f, we evaluate f at the endpoints $x = -1, x = 8$ and at the critical point $x = 0$. Since $f(-1) = 1, f(0) = 0, f(8) = 4$, then $f(0) = 0$ is an absolute minimum and $f(8) = 4$ is an absolute maximum.

Theorem 4.2.2 *If f is differentiable and has an extreme value at an interior point c then $f'(c) = 0$.*

If $f'(c) = 0$, this does not mean that f has an extreme value (maximum or minimum) at $x = c$. For example, $x = 0$ is a critical point of $f(x) = x^3$ but $f(0)$ is neither maximum nor minimum for $y = x^3$.

To classify the critical as maximum or minimum, we can use either the first derivative test or the second derivative test which we state now.

Theorem 4.2.3 ***(First derivative test)*** *Suppose that f has a critical point at $c = c$ and that $f'(x)$ exists in an open interval containing $x = c$. Then*

(a) If f' changes sign from positive to negative at $x = c$ then $f(c)$ is a local maximum.

(b) If f' changes sign from negative to positive at $x = c$ then $f(c)$ is a local minimum.

(c) If f' does not change sign at $x = c$ then f does not have an extreme value at $x = c$.

Theorem 4.2.4 ***(Second derivative test)*** *Suppose that $f'(c) = 0$ and that f'' is continuous in an open interval containing c. Then*

(a) If $f''(c) < 0$ then $f(c)$ is a local maximum.

(b) If $f''(c) > 0$ then $f(c)$ is a local minimum.

(c) If $f''(c) = 0$ then the test fails.

If $f''(x) \geq 0$ for all x in an interval I then f is concave up on I. If $f''(x) \leq 0$ for all x in an interval I then f is concave down on I.

Definition 4.2.3 *A point where f has tangent line and changes concavity is called* ***an inflection point*** *of f.*

Example 4.2.3 Find the intervals at which the function $f(x) = x^4 - 4x^3 + 10$ is increasing, decreasing, concave up and concave down. Then, find the extreme values of f. Notice that $f'(x) = 4x^2(x - 3)$ and $f'(x) = 0$ at $x = 0, 3$, $f' < 0$ on $(-\infty, 0) \cup (0, 3)$ (f is decreasing) and $f' > 0$ on $(3, \infty)$ (f is increasing). It follows that $f(3) = -17$ is an absolute minimum.

Now, $f''(x) = 12x(x-2)$, from which we conclude that $f''(x) = 0$ at $x = 0, 2$. Moreover, $f''(x) > 0$ on $(-\infty, 0) \cup (2, \infty)$ (f is concave up) and $f''(x) < 0$ on $(0, 2)$ (concave down).

f has inflection points at $(0, 10)$ and $(2, -6)$.

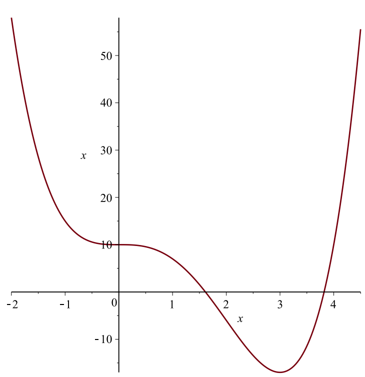

Figure 4.1: Graph of $y = x^4 - 4x^3 + 10$

4.3 The Mean Value Theorem

Theorem 4.3.1 ***Rolle's Theorem*** *If $y = f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on (a, b) and $f(a) = f(b)$, then there is at least one point c in (a, b) such that $f'(c) = 0$.*

Theorem 4.3.2 ***The Mean Values Theorem*** *If $y = f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on (a, b), then there is at least one point c in (a, b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example 4.3.1 Let $f(x) = x^2$, $x \in [1, 4]$. Then

$$\frac{f(4) - f(1)}{4 - 1} = \frac{15}{3} = 2c \quad \text{so} \quad c = \frac{5}{2}$$

4.4 Exercises

1. Find the intervals in which the following functions are increasing, decreasing, concave up and concave down. Then, find the extreme values and inflection points and sketch their graphs:

   (a) $y = 1 - (x + 1)^3$

   (b) $y = \frac{x^2+1}{x}$

   (c) $y = x^4 - 2x^2$

   (d) $y = \frac{x}{x^2+1}$

   (e) $y = \frac{x^2-3}{x-2}$

   (f) $y = \sqrt[3]{x^3 + 1}$

   (g) $y = \frac{x}{x^2-1}$

   (h) $y = x\sqrt{8 - x^2}$

2. Find the value of c in the conclusion of the mean value theorem for the function $f(x) = \sqrt{x}$ on the interval $[a, b]$, $a > 0$.

3. For what values of a, m and b does the function

$$f(x) = \begin{cases} 3 & , \quad x = 0 \\ -x^2 + 3x + a & , \quad 0 < x < 1 \\ mx + b & , \quad 1 \le x \le 2 \end{cases}$$

   satisfy the hypotheses of the mean value theorem on the interval $[0, 2]$.

Chapter 5

Integration

[1]

5.1 Antiderivative and integration

Definition 5.1.1 *A function F is called an **antiderivative** of a function f on an interval I if $F'(x) = f(x)$, for all x in I. The set of all antiderivatives of f is called the **indefinite integral** of f and is denoted by $\int f(x)dx$.*

Example 5.1.1 In this example, we give the indefinite integrals of some important functions

(a) $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$

(b) $\int \sin x dx = -\cos x + C$

(c) $\int \cos x dx = \sin x + C$

(d) $\int \sec^2 x dx = \tan x + C$

(e) $\int \sec x \tan x dx = \sec x + C$

(f) $\int \csc x \cot x dx = -\csc x + C$

(g) $\int \csc^2 x dx = -\cot x + C$

Example 5.1.2 Consider the following examples:

(a) $\int (x^{-2} - x^2 + 1)dx = -\frac{1}{x} - \frac{1}{3}x^3 + x + C$

(b) $\int \cos^2 \theta d\theta = \int \frac{1+\cos(2\theta)}{2} d\theta = \frac{\theta}{2} + \frac{\sin(2\theta)}{4} + C$

[1] This part is a review of chapter 5 of the textbook

5.2 Definite integrals and areas

Sometimes, we evaluate integrals on given intervals. Such integrals are called definite integrals and take the form

$$\int_a^b f(x)dx$$

We can solve definite integrals using the fundamental theorem of calculus:

Theorem 5.2.1 ***Fundamental Theorem of Calculus***

(I) Suppose that f is continuous on $[a,b]$ and F is an is an antiderivative of f on $[a,b]$ then

$$\int_a^b f(x)dx = F(b) - F(a)$$

(II) Suppose that f is continuous on $[a,b]$ and $F(x) = \int_a^x f(t)dt$ then F is continuous on $[a,b]$ and differentiable on (a,b) and $F'(x) = f(x)$.

If $f(x) \geq 0$ is an integrable function on $[a,b]$ then $\int_a^b f(x)dx$ is the area enclosed between the curve $f(x)$ and the $x-$axis.

Example 5.2.1 Find the derivatives of the following functions

(a) $\frac{d}{dx}\int_0^x \sin t dt = \sin x$.

(b) $\frac{d}{dx}\int_1^{x^2} \frac{dt}{1+t^2} = \frac{2x}{1+x^4}$

Example 5.2.2 Find the area enclosed between the following curves and the $x-$axis in the given intervals

(a) $f(x) = 2x\sqrt{x^2+1}$, $x \in [0,1]$. The area is given by the following integral that we solve using substitution $u = x^2 + 1$

$$A = \int_0^1 2x\sqrt{x^2+1}dx = \int_1^2 u^{1/2}du = \frac{2}{3}u^{3/2}|_1^2 = \frac{2}{3}(2\sqrt{2}-1)$$

We can find the area enclosed between two functions $f(x)$ and $g(x)$ in some interval $[a,b]$ where $f(x) \geq g(x)$, using the formula

$$A = \int_a^b (f(x) - g(x))dx$$

Sometimes, the functions are expressed in terms of y in some interval $[c,d]$, so the area in this case is

$$A = \int_c^d (f(y) - g(y))dy$$

The next examples explain both cases.

Example 5.2.3 Find the area enclosed between the curves $f(x) = 2 - x^2$ and $y = -x$.
Solution We first find the points at which the two curves intersect by equating the functions

$$-x = 2 - x^2 \quad \text{which is equivalent to} \quad x^2 - x - 2 = 0$$

The last equation can be factorized as $(x+1)(x-2) = 0$. Thus, the two curves intersect at $x = -1$ and $x = 2$. So, the area is given by

$$A = \int_{-1}^{2} (2 - x^2 + x)dx = \frac{9}{2}$$

Example 5.2.4 Find the area enclosed between the curves $y = \sqrt{x}$, the $x-$axis and the line $y = x - 2$. It is easier to write x as a function of y and to integrate with respect to y. In this case, we have $x = y^2$ and $x = y + 2$. The two curves intersect at the point $y = 2$. The area is given by the integral

$$A = \int_{0}^{2} (y + 2 - y^2)dy = \frac{10}{3}$$

5.3 Exercises

1. Solve the following integrals:
   (a) $\int \sin(5x)dx$
   (b) $\int \tan^2 x dx$
   (c) $\int (1 + \cot^2 \theta)d\theta$.
   (d) $\int \frac{\csc \theta d\theta}{\csc \theta - \sin \theta}$
2. Find the derivatives of the following functions
   (a) $y = \int_1^x \frac{dt}{t}$
   (b) $y = \int_0^{\sqrt{x}} \cos t dt$
   (c) $y = \int_{\tan x}^0 \frac{dt}{1+t^2}$
3. Find the linearization of $g(x) = 3 + \int_1^{x^2} \sec(t-1)dt$ at $x = -1$
4. Solve the following definite integrals
   (a) $\int_1^{\sqrt{2}} \frac{s^2+\sqrt{s}}{s^2} ds$
   (b) $\int_0^{\pi/6} (\sec x + \tan x)^2 dx$
   (c) $\int_0^{\pi} (\cos x + |\cos x|)dx$

5. Use substitution to solve the following integrals:

   (a) $\int \frac{dx}{\sqrt{x}(1+\sqrt{x})^2}$

   (b) $\int \frac{\sec z \tan z}{\sqrt{\sec z}} dz$

   (c) $\int \sqrt{\frac{x-1}{x^5}} dx$

   (d) $\int x^3 \sqrt{x^2+1} dx$

6. Find the area enclosed between the given functions:

   (a) $y = x^2 - 2x$, $y = x$

   (b) $y = x^2$, $y = -x^2 + 4x$

   (c) $x = y^2$, $x = 3 - 2y^2$

   (d) $x = y^3 - y^2$, $x = 2y$