Math141-Calculus I: Review of differentiation and integration Lecture notes based on Thomas Calculus Book Chapter 1 to Chapter 5

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Chapter 1

Functions

[1]

1.1 Functions

In this lecture, we review some important functions with their domains, ranges and graphs.

Definition 1.1.1 *A function f is a rule that assigns to each point x in the domain a unique point $y = f(x)$ in the range of f. We write $f : D \to R$ where D is the domain of f and R is its range.*

Remark 1.1.1 The set of x-values at which $f(x)$ is defined forms the domain of f while the set of $y-$values (the set of the images of the $x-$values) forms the range of f. The domain of x appears on the horizontal axis (the $x-$axis), while the range of f appears on the vertical axis (the $y-$axis).

Now, we give some important basic functions with their domains, ranges and graphs.

[1] This part is a review of chapter 1 in the textbook

Example 1.1.1 (a) $f(x) = x^2$, $D = (-\infty, \infty)$, $R = [0, \infty)$. If we let $y = x^2$ then $x \in (-\infty, \infty)$, $y \in [0, \infty)$.

(b) $f(x) = \sqrt{x}$, $D = R = [0, \infty)$, hence $x, y \in [0, \infty)$.

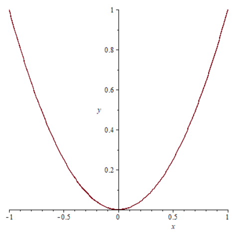


Figure 1.1: Graph of $y = x^2$

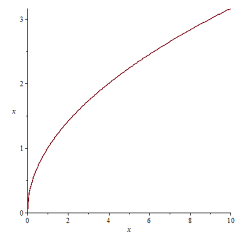


Figure 1.2: Graph of $y = \sqrt{x}$

(c) The absolute value function $f(x) = |x| = \sqrt{x^2}$, $D = (-\infty, \infty)$, $R = [0, \infty)$. Then, $x \in (-\infty, \infty)$, $y \in [0, \infty)$.

(d) $f(x) = \sqrt{1 - x^2}$. The domain of f is the set of values of x such that $1 - x^2 \geq 0$, so we must have $x^2 \leq 1$. Taking the square root of both sides, we get $\sqrt{x^2} \leq 1$ which implies that $|x| \leq 1$. The last inequality is equivalent to $-1 \leq x \leq 1$. We find that $x \in [-1, 1]$, $y \in [0, 1]$. So, $D = [-1, 1], R = [0, 1]$.

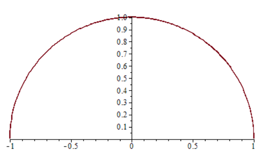


Figure 1.3: Graph of $y = \sqrt{1 - x^2}$

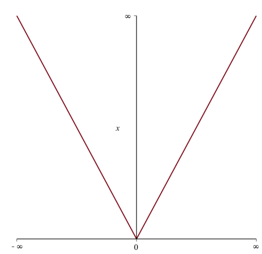


Figure 1.4: Graph of $y = |x|$

(e) The greatest integer function $f(x) = \lfloor x \rfloor$, $D = (-\infty, \infty)$, $R = 0, \pm 1, \pm 2, ...$

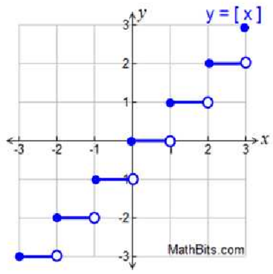


Figure 1.5: Graph of $y = \lfloor x \rfloor$

1.2 Trigonometric functions

In this section, we review the six trigonometric functions: $\sin x, \cos x, \tan x, \cot x$, $\sec x$ and $\csc x$. You are supposed to know the values of these functions at the main values $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \ldots$

(a) $y = \sin x,\ D = (-\infty, \infty),\ R = [-1, 1]$.

(b) $y = \cos x,\ D = (-\infty, \infty),\ R = [-1, 1]$.

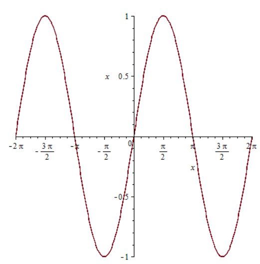


Figure 1.6: Graph of $y = \sin x$

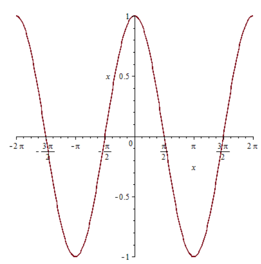


Figure 1.7: Graph of $y = \cos x$

Note that

$$\boxed{\cos x = 0 \text{ if } x = \frac{\pi}{2} \pm n\pi \text{ and } \sin x = 0 \text{ if } x = \pm n\pi,\ n = 0, 1, 2, \ldots}$$

(c) $y = \tan x = \frac{\sin x}{\cos x},\ D = (-\infty, \infty) \setminus \{\frac{\pi}{2} \pm n\pi\},\ n = 0, 1, 2, \ldots,\ R = (-\infty, \infty)$

(d) $y = \cot x = \frac{\cos x}{\sin x}$, $D = (-\infty, \infty) \setminus \{\pm n\pi\}$, $n = 0, 1, 2, ...$, $R = (-\infty, \infty)$

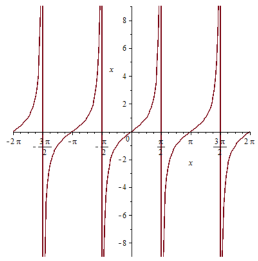


Figure 1.8: Graph of $y = \tan x$

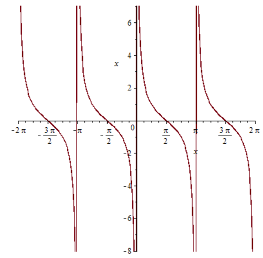


Figure 1.9: Graph of $y = \cot x$

(e) $y = \sec x = \frac{1}{\cos x}$, $D = (-\infty, \infty) \setminus \{\frac{\pi}{2} \pm n\pi\}$, $n = 0, 1, 2, ...$,
$R = (-\infty, -1] \cup [1, \infty)$

(f) $y = \csc x = \frac{1}{\sin x}$, $D = (-\infty, \infty) \setminus \{\pm n\pi\}$, $n = 0, 1, 2, ...$,
$R = (-\infty, -1] \cup [1, \infty)$

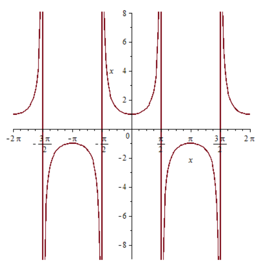


Figure 1.10: Graph of $y = \sec x$

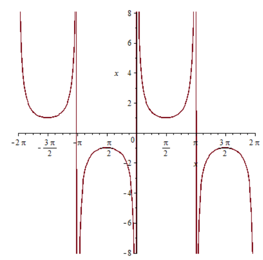


Figure 1.11: Graph of $y = \csc x$

Remark 1.2.1 We have the following results

- Since $\sin(x + 2\pi) = \sin x$, $\cos(x + 2\pi) = \cos x$, $\sec(x + 2\pi) = \sec x$ and $\csc(x + 2\pi) = \csc x$, the functions $\sin x, \cos x, \sec x$ and $\csc x$ are called periodic with period 2π.

- Since $\tan(x+\pi) = \tan x$ and $\cot(x+\pi) = \cot x$ then $\tan x$ and $\cot x$ are periodic with period π.

1.2.1 Trigonometric identities

1. $\sin^2 x + \cos^2 x = 1$.
2. $\sin(2x) = 2\sin x\cos x$.
3. $\cos(2x) = \cos^2 x - \sin^2 x$.
4. $\cos^2 x = \frac{1+\cos(2x)}{2}$.
5. $\sin^2 x = \frac{1-\cos(2x)}{2}$.
6. $\sec^2 x = 1 + \tan^2 x$.
7. $\csc^2 x = 1 + \cot^2 x$.
8. $\cos(A+B) = \cos A\cos B - \sin A\sin B$.
9. $\sin(A+B) = \sin A\cos B + \cos A\sin B$.

Example 1.2.1 Using the above identities, we find the following:

(a) $\sin(x+\pi) = \sin(x)\underbrace{\cos(\pi)}_{\text{-1}} + \cos(x)\underbrace{\sin(\pi)}_{0} = -\sin x,$

(b) $\cos(x+\pi) = \cos(x)\underbrace{\cos(\pi)}_{\text{-1}} - \sin(x)\underbrace{\sin(\pi)}_{0} = -\cos x.$

(c) $\sin(x+\frac{\pi}{2}) = \sin(x)\underbrace{\cos(\frac{\pi}{2})}_{0} + \cos(x)\underbrace{\sin(\frac{\pi}{2})}_{1} = \cos x,$

(d) $\cos(x+\frac{\pi}{2}) = \cos(x)\underbrace{\cos(\frac{\pi}{2})}_{0} - \sin(x)\underbrace{\sin(\frac{\pi}{2})}_{1} = -\sin x$

1.3 Even and odd functions

Definition 1.3.1 *Let f be a function defined on an interval $I = [-a, a]$, where a is a positive real number. Then*

- *$f(x)$ is called even if $f(-x) = f(x)$. If f is even then its graph is symmetric about the $y-$axis.*
- *$f(x)$ is called odd if $f(-x) = -f(x)$. If f is odd then its graph is symmetric about the origin.*

Example 1.3.1 $x^2, x^4, x^6, ..., \cos x, \sec x$ are even functions. $x, x^3, x^5, ...,$ $\sin x, \tan x, \csc x, \cot x$ are odd functions.

Example 1.3.2 Determine whether the functions $f(x) = x^2 + |x|$, $g(x) = x^3 + x^5$, $h(x) = x + x^2$ are even, odd or neither.

$$f(-x) = (-x)^2 + |-x| = x^2 + |x| = f(x), \quad \text{so } \ f \ \text{ is even}$$

$g(-x) = (-x)^3 + (-x)^5 = -x^3 - x^5 = -(x^3 + x^5) = -g(x)$ so g is odd

$h(-x) = (-x) + (-x)^2 = -x + x^2$ then $h(-x) \neq h(x)$, $h(-x) \neq -h(x)$

we conclude that h is neither even nor odd.

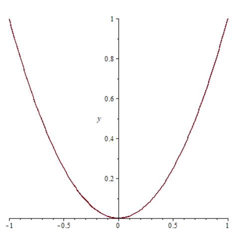


Figure 1.12: Graph of $y = x^2$

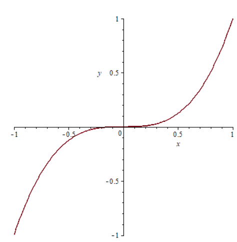


Figure 1.13: Graph of $y = x^3$

1.4 Exercises

(1) Find the domain and the range of the following functions:

(a) $f(x) = \frac{1}{\sqrt{x}}$.

(b) $f(x) = \tan(\pi x)$.

(c) $f(x) = 1 + |x|$.

(d) $f(x) = \sec^2 x$.

(e) $g(x) = \frac{1}{x^2}$.

(f) $h(x) = \frac{1}{\sqrt{1-x^2}}$.

(2) Sketch the following functions:

(a) $y = \sin(\pi x)$

(b) $y = |x - 1|$

(c) $y = \cos(x) + 1$

(3) Determine whether the following functions are even, odd or neither:

(a) $f(x) = x^2 + 1$.

(b) $f(x) = x^3 + x$.

(c) $g(t) = \frac{1}{t-1}$.

(d) $h(x) = \frac{x}{x^2-1}$.

(4) Prove the following:

(a) If $f(x)$ is even and $g(x)$ is odd then $(g \circ f)(x)$ is even.

(b) If $f(x)$ is even and $g(x)$ is odd then $\frac{f(x)}{g(x)}$ is odd.

Chapter 2

Limits and continuity

[1]

2.1 Limits of functions

When a function f approaches a certain limit L as x approaches a, we write

$$\lim_{x \to a} f(x) = L$$

This limit means that *the function gets arbitrarily close to L when x is sufficiently close to a.* Notice that a or L or both of them can be $+\infty$ or $-\infty$. The function f may or may not be defined at $x = a$. As you know,

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$$

where $\lim_{x \to a^+} f(x)$ is the limit of $f(x)$ as x approaches a from the right (also called the right-hand limit) and $\lim_{x \to a^-} f(x)$ is the limit of $f(x)$ as x approaches a from the left (also called the left-hand limit).

[1] This is a review of chapter two in the textbook

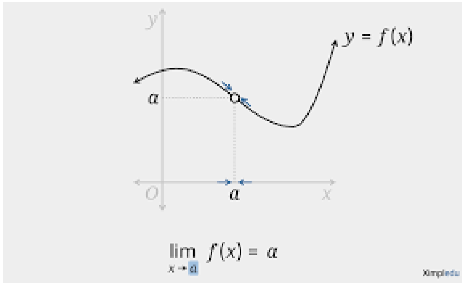


Figure 2.1: Limit of a function

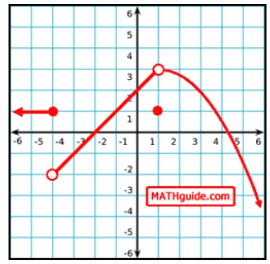


Figure 2.2: Example of limits

Example 2.1.1 We can use simple techniques to find the following limits:

(a) $\lim\limits_{x\to 1} \frac{x-1}{x+1} = 0.$

(b) $\lim\limits_{x\to 1} \frac{x^2-1}{x-1} = \lim\limits_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = 2.$

(c) $\lim\limits_{x\to +\infty} \frac{1}{x} = 0.$

(d) $\lim\limits_{x\to 0^+} \frac{1}{x} = +\infty.$

(e) $\lim\limits_{x\to 1} \frac{x^2+x-2}{x^2-x} = \lim\limits_{x\to 1} \frac{(x+2)\cancel{(x-1)}}{x\cancel{(x-1)}} = 3.$

(f) $\lim\limits_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} = \lim\limits_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} \frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3}$
$(\sqrt{x^2+8}-3)(\sqrt{x^2+8}+3) = (\sqrt{x^2+8})^2 - 3\sqrt{x^2+8} + 3\sqrt{x^2+8} - 9 = x^2+8-9 = x^2-1 = (x-1)(x+1)$
$\lim\limits_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} = \lim\limits_{x\to -1} \frac{(x-1)\cancel{(x+1)}}{\cancel{(x+1)}\sqrt{x^2+8}+3} = \frac{-2}{6} = -\frac{1}{3}.$

Theorem 2.1.1 *(**The Sandwich Theorem**) Suppose that*

$$g(x) \le f(x) \le h(x)$$

for all x in some open interval containing c, except possibly at $x = c$ and that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L \quad \textit{then} \quad \lim_{x\to c} f(x) = L$$

Example 2.1.2 Suppose that $f(x)$ is a function that satisfies

$$1 - x^2 \leq f(x) \leq 1 + x^2$$

Then $\lim_{x\to 0} f(x) = 1$ since $\lim_{x\to 0}(1 - x^2) = \lim_{x\to 0}(1 + x^2) = 1$.

Example 2.1.3 Find $\lim_{x\to +\infty} \frac{\sin x}{x}$. Since

$$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

and $\lim_{x\to\infty} \frac{1}{x} = 0$, then, by the sandwich theorem

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0$$

Remark 2.1.1 Please do not confound the previous limit with $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Example 2.1.4 Consider the function

$$f(x) = \begin{cases} x + 1 & , \quad x \leq 0 \\ -x & , \quad x > 0 \end{cases}$$

Then, $\lim_{x\to 0^+} f(x) = 0$ and $\lim_{x\to 0^-} f(x) = 1$. So, $\lim_{x\to 0} f(x)$ does not exist.

We give another example

Example 2.1.5 Consider the function

$$g(x) = \begin{cases} x + 2 & , \quad x \leq -1 \\ x^2 & , \quad -1 < x \leq 1 \\ x - 1 & , \quad x > 1 \end{cases}$$

Then, $\lim_{x\to -1^+} g(x) = \lim_{x\to -1^-} g(x) = 1$, so $\lim_{x\to -1} g(x) = 1$. While, $\lim_{x\to 1^+} g(x) = 0$, $\lim_{x\to 1^-} g(x) = 1$, so $\lim_{x\to 1} g(x)$ does not exist.

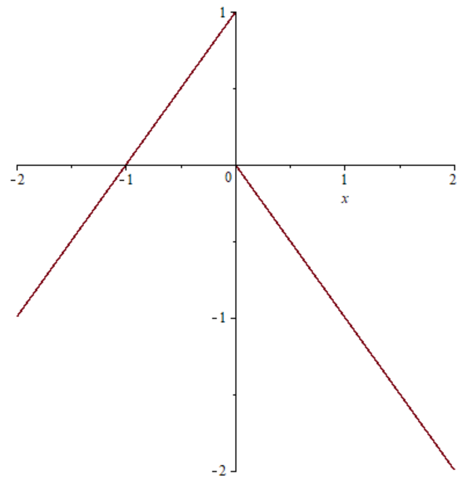


Figure 2.3: Graph of $f(x)$

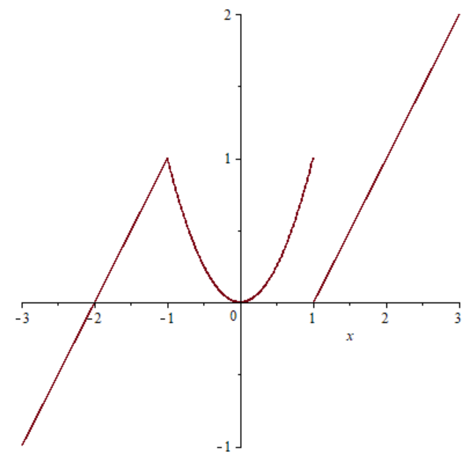


Figure 2.4: Graph of $g(x)$

2.2 Continuity

Definition 2.2.1 *A function f is continuous at a point x_0 if the following conditions are satisfied:*

(a) $f(x_0)$ *exists.*

(b) $\lim_{x\to x_0} f(x)$ *exists.*

(c) $\lim_{x\to x_0} f(x) = f(x_0)$.

Example 2.2.1 The functions $\sin x, \cos x$, $|x|$ and all polynomials are continuous on $(-\infty, \infty)$.

Example 2.2.2 The rational functions are continuous at all points except at the zeros of the denominator. For example, the function

$$f(x) = \frac{x^3 + x + 1}{x^2 - 1}$$

is continuous on $(-\infty, \infty) \setminus \{-1, 1\}$.

Example 2.2.3 **(a function with removable discontinuity)** Consider the function

$$f(x) = \frac{x^2 + 2x - 3}{x^2 - 1}$$

Then

$$\lim_{x \to 1} \frac{x^2 + 2x - 3}{x^2 - 1} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x+3)}{\cancel{(x-1)}(x+1)} = \lim_{x \to 1} \frac{x+3}{x+1} = 2$$

The point $x = 1$ is called a **removable discontinuity** of the function f because we can define f at $x = 1$ so that we can remove the discontinuity. The following function is called the **continuous extension of f at $x = 1$**

$$F(x) = \begin{cases} f(x) & , \quad x \neq 1 \\ 2 & , \quad x = 1 \end{cases}$$

Theorem 2.2.1 ***(The intermediate value theorem)*** *If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.*

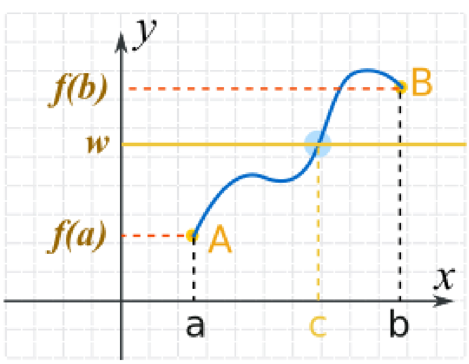


Figure 2.5: Intermediate Value Theorem

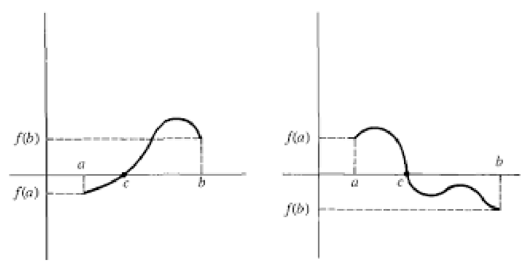


Figure 2.6: Graph of $g(x)$

Recall that a point c is called a root of a function f if $f(c) = 0$. We can use the intermediate value theorem to show that a given function has a root in some interval **(Bolzano Theorem)**.

Example 2.2.4 Consider the function $f(x) = x^3 - x - 1$. Take $a = 1$ and $b = 2$. Since $f(1) = -1 < 0$, $f(2) = 5 > 0$ and $f(1) < 0 < f(2)$ then there exists $c \in [1, 2]$ such that $f(c) = 0$. In fact, $c \approx 1.324717957$.

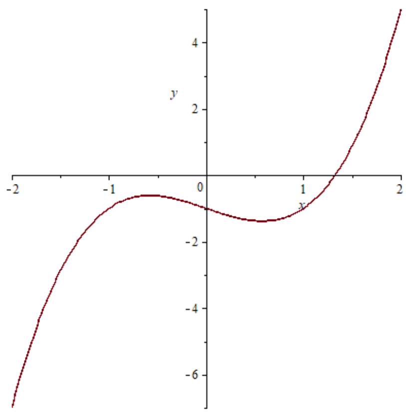


Figure 2.7: Graph of $y = x^3 - x - 1$

2.2.1 Asymptotes

In this section, we are dealing mainly with rational functions. **A rational function is the ratio of two polynomials**. Our objective is to be able to sketch some rational functions using limits and asymptotes. *A method that helps us in finding the limits of a rational function as x approaches $+\infty$ or $-\infty$, we divide the numerator and denominator by the highest power in the denominator.* Suppose that we want to find the limits of a rational function

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$ is a polynomial of degree m and $q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$ is a polynomial of degree n. Then, we have the following cases:

(a) if $m = n$ then $\lim_{x\to\pm\infty} f(x) = \frac{a_m}{b_n}$. For example, $\lim_{x\to\pm\infty} \frac{2x^3-x+3}{3x^3+x^2+x} = \frac{2}{3}$

(b) if $m < n$ then $\lim_{x\to\pm\infty} f(x) = 0$. For example, $\lim_{x\to\pm\infty} \frac{x^2+1}{x^3+x} = 0$

(c) if $m > n$ then $\lim_{x\to\pm\infty} f(x) = \pm\infty$. For example, to find $\lim_{x\to\infty} \frac{x^2+1}{x+1}$, we divide the numerator and denominator by x to get $\lim_{x\to\infty} \frac{x+\frac{1}{x}}{1+\frac{1}{x}} = +\infty$

Definition 2.2.2 *A line $y = b$ is a horizontal asymptote of the graph of the function $y = f(x)$ if either*

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b$$

Example 2.2.5 The line $y = 0$ is a horizontal asymptote for graph of the function $f(x) = \frac{x}{x^2+1}$ since $\lim_{x\to+\infty} \frac{x}{x^2+1} = \lim_{x\to-\infty} \frac{x}{x^2+1} = 0$.

Example 2.2.6 The line $y = 1$ is a horizontal asymptote for the graph of the function $f(x) = \frac{x^2}{x^2+1}$ since $\lim_{x\to+\infty} \frac{x^2}{x^2+1} = \lim_{x\to-\infty} \frac{x^2}{x^2+1} = 1$.

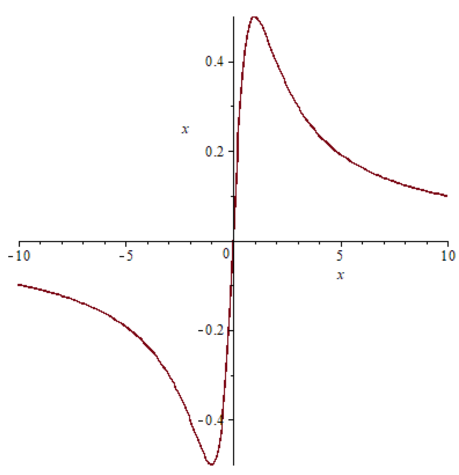


Figure 2.8: Graph of $f(x) = \frac{x}{x^2+1}$

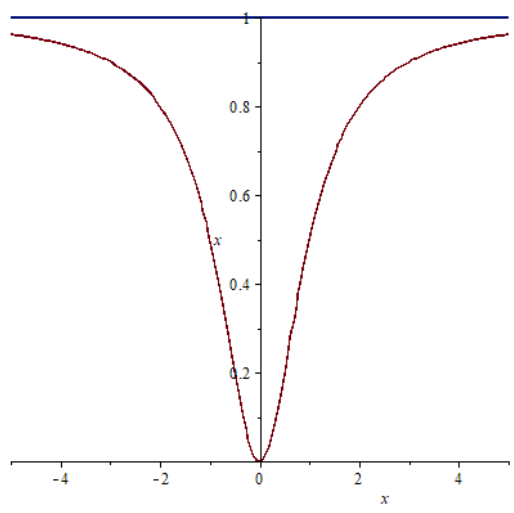


Figure 2.9: Graph of $f(x) = \frac{x^2}{x^2+1}$

Definition 2.2.3 *A line $x = a$ is a vertical asymptote of the graph of the function $y = f(x)$ if either*

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \textit{or} \quad \lim_{x\to a^-} f(x) = \pm\infty$$

Example 2.2.7 The line $x = 0$ is a vertical asymptote for $f(x) = \frac{1}{x}$ since $\lim\limits_{x\to 0^+} \frac{1}{x} = +\infty$ and $\lim\limits_{x\to 0^-} \frac{1}{x} = -\infty$.

Example 2.2.8 Consider the function $f(x) = \frac{x+1}{x-1}$. Notice that

$$\lim_{x\to 1^+} \frac{x+1}{x-1} = +\infty, \quad \lim_{x\to 1^-} \frac{x+1}{x-1} = -\infty$$

and

$$\lim_{x\to +\infty} \frac{x+1}{x-1} = \lim_{x\to -\infty} \frac{x+1}{x-1} = 1$$

Then the line $x = 1$ is a vertical asymptote and the line $y = 1$ is a horizontal asymptote.

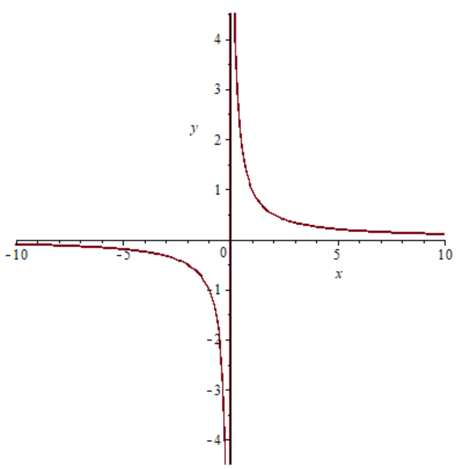


Figure 2.10: Graph of $f(x) = \frac{1}{x}$

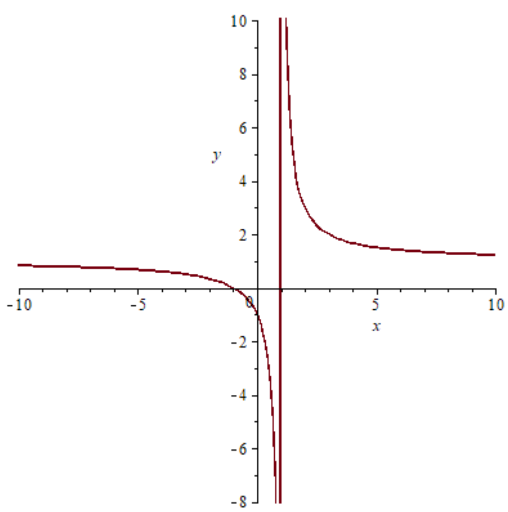


Figure 2.11: Graph of $f(x) = \frac{x+1}{x-1}$

Consider the following remarks:

Remark 2.2.1 Suppose that $f(x)$ is a rational function

(a) the graph of $f(x)$ can intersect its horizontal asymptote as in example (2.2.6).

(b) the graph of $f(x)$ can have horizontal and vertical asymptotes.

(c) the graph of f can have at most one horizontal asymptote.

(d) $x = a$ is a vertical asymptote for the graph of f if $x = a$ is a root of the denominator of f. But if $x = a$ is a root of the denominator of f then the graph of f does not have necessarily a vertical asymptote at $x = a$. For example, the graph of the function $f(x) = \frac{x^2+2x-3}{x^2-1}$ does not have a vertical asymptote at $x = 1$, see example (2.2.3). Also, the graph of the function $f(x) = \frac{\sin x}{x}$, which is not a rational function, does not have a vertical asymptote at $x = 0$.

Example 2.2.9 The function $f(x) = \frac{\sin x}{x}$ has no vertical asymptote even it is undefined at $x = 0$ since $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$.

Example 2.2.10 Let $f(x) = \frac{x^2+2x-3}{x^2-1}$, see example(2.2.3)

$$\lim_{x\to 1} \frac{x^2+2x-3}{x^2-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x+3)}{\cancel{(x-1)}(x+1)} = \lim_{x\to 1} \frac{x+3}{x+1} = 2$$

and

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \frac{x+3}{x+1} = +\infty$$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{x+3}{x+1} = -\infty$$

from the previous limits, we conclude that $x = -1$ is a vertical asymptote but $x = 1$ is not a vertical asymptote.

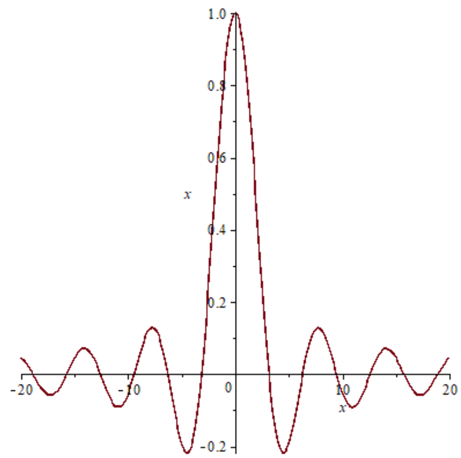


Figure 2.12: Graph of $f(x) = \frac{\sin x}{x}$

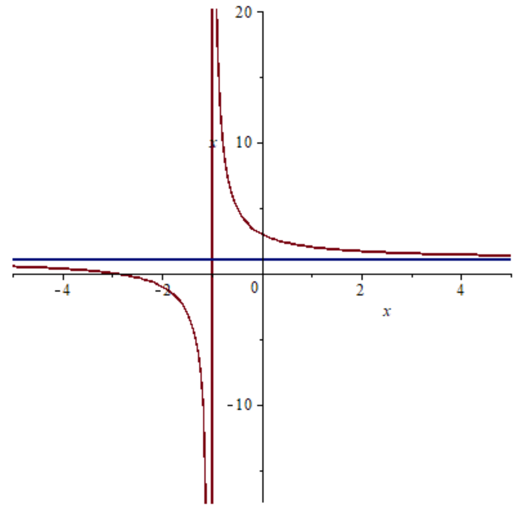


Figure 2.13: Graph of $f(x) = \frac{x^2+2x-3}{x^2-1}$

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator then the graph of f has an **oblique asymptote**.

Example 2.2.11 The graph of the function $f(x) = \frac{x^2}{x-1}$ has an oblique asymptote since the degree of the numerator is 2 and the degree of the denominator is one. Using polynomial division, we can write

$$f(x) = (x+1) + \frac{1}{x-1}$$

So, the line $y = x + 1$ is the oblique asymptote of the graph of f. Moreover, the line $x = 1$ is a vertical asymptote for the graph of f since $\lim_{x \to 1^+} f(x) = +\infty$ and $\lim_{x \to 1^-} f(x) = -\infty$. *Note that a rational function cannot have a horizontal and an ablique asymptote at the same time.*

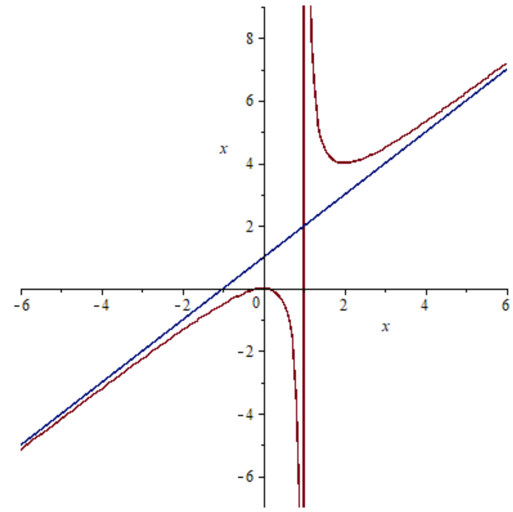


Figure 2.14: Graph of $y = \frac{x^2}{x-1}$

2.3 Exercises

1. Find the following limits:

   (a) $\lim\limits_{t\to -1} \frac{t^2+3t+2}{t^2-t-2}$

   (b) $\lim\limits_{x\to 1} \frac{1-\sqrt{x}}{1-x}$

   (c) $\lim\limits_{\theta\to 1} \frac{\theta^4-1}{\theta^3-1}$

   (d) $\lim\limits_{\theta\to 0} \frac{\sin(2\theta)}{3\theta}$

   (e) $\lim\limits_{\theta\to 0} \frac{1-\cos\theta}{\sin(2\theta)}$

   (f) $\lim\limits_{x\to \infty} \frac{1+\sqrt{x}}{1-\sqrt{x}}$

   (g) $\lim\limits_{x\to -\infty} \frac{\sqrt{x^2+1}}{x+1}$

   (h) $\lim\limits_{x\to -\infty} \frac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}}$

   (i) $\lim\limits_{x\to \infty} (\sqrt{x^2+1} - \sqrt{x^2-x})$

(j) $\lim_{t\to 3^+} \frac{\lfloor t \rfloor}{t}$

(k) $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$

2. Find the asymptotes of the following functions then sketch their graphs

   (a) $f(x) = \frac{x+1}{x-1}$

   (b) $y = \frac{x^3+1}{x^2}$

   (c) $f(x) = \frac{x^2+1}{x-1}$

   (d) $f(x) = \frac{x^3+1}{x^2-1}$

3. For what values of a and b is

$$g(x) = \begin{cases} ax + 2b & , \quad x \le 0 \\ x^2 + 3a - b & , \quad 0 < x \le 2 \\ 3x - 5 & , \quad x > 2 \end{cases}$$

   continuous at every x. Then sketch the graph of the function.

4. Find the continuous extension of the function $h(t) = \frac{t^2+3t-10}{t-2}$.

5. Use the intermediate value theorem to show that the function $f(x) = x^3 - 2x^2 + 2$ has a root.

Chapter 3

Differentiation

3.1 Definition of derivative

Definition 3.1.1 *The derivative of a function f at x_0, denoted $f'(x_0)$ is defined by*

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists.

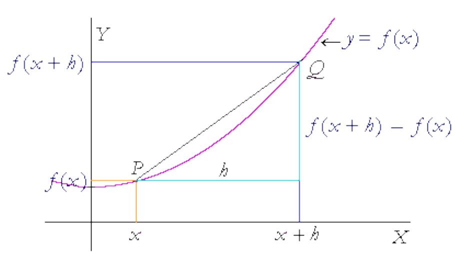


Figure 3.1: Secant line

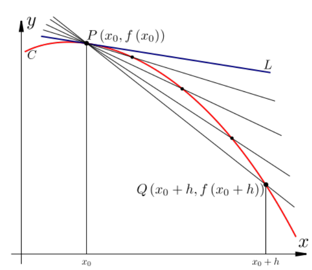


Figure 3.2: Tangent line

Let $z = x_0 + h$, then $h = z - x_0$. The above limit can be written as

$$f'(x_0) = \lim_{z \to x_0} \frac{f(z) - f(x_0)}{z - x_0}$$

If $f'(x_0)$ exists then we say that f is **differentiable** at x_0. We say that f is differentiable on an open interval (a, b) if it is differentiable at each point of (a, b). We can use the above definition to find the derivative of any differentiable function at any point. The derivative of f at x_0 gives the rate of change of f at x_0. It is also the slope of the tangent line to the graph of f at $(x_0, f(x_0))$.

Example 3.1.1 Use the definition to find the derivative of the function $f(x) = \sqrt{x}$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

When we say that f is differentiable on a closed interval $[a, b]$, we mean the following

- f' exists at all points in the open interval (a, b).
- The **right-hand derivative of f at** a exists; that is,

$$f'_+(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

exists. We denote the right-hand derivative of f at $x = a$ by $f'_+(a)$.

- The **left-hand derivative of f at** b exists; that is,

$$f'_-(b) = \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$$

exists. We denote the left-hand derivative of f at $x = b$ by $f'_-(b)$.

Remark 3.1.1 A function f is differentiable at $x = c$ if and only if the right-hand derivative and the left-hand derivative both exist and are equal at $x = c$.

If f is differentiable at $x = c$ then f is continuous at $x = c$. The converse of this statement is not true, the function $f(x) = |x|$ is continuous but not differentiable at $x = 0$.

Example 3.1.2 Let $f(x) = |x|$. We find the left-hand and right-hand derivatives of f at $x = 0$.

$$f'_+(0) = \lim_{h \to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$

$$f'_-(0) = \lim_{h \to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$$

We conclude that f is not differentiable at $x = 0$.

Example 3.1.3 Determine whether the following function is differentiable at $x = 0$

$$f(x) = \begin{cases} x^{2/3} & , \quad x \geq 0 \\ x^{1/3} & , \quad x < 0 \end{cases}$$

Using the definition of the derivative

$$f'_+(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h^{2/3}}{h} = \lim_{h \to 0^+} \frac{1}{h^{1/3}} = +\infty$$

$$f'_-(0) = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{h^{1/3}}{h} = \lim_{h \to 0^-} \frac{1}{h^{2/3}} = +\infty$$

So, f is not differentiable at $x = 0$. The graph of $f(x)$ has a vertical tangent at $x = 0$.

3.2 Differentiation rules

Theorem 3.2.1 *Suppose that $f(x)$ and $g(x)$ are differentiable at x, c is a constant. Then*

(1) $\frac{d}{dx}(c) = 0$

(2) $\frac{d}{dx}x^n = nx^{n-1}$, *where n is a positive integer.*

(3) $\frac{d}{dx}(cf(x)) = c\frac{df}{dx}$

(4) $\frac{d}{dx}(f(x) \pm g(x)) = \frac{df}{dx} \pm \frac{dg}{dx}$

(5) $\frac{d}{dx}(f(x)g(x)) = \frac{df}{dx}g(x) + f(x)\frac{dg}{dx}$

(6) $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{df}{dx} - f(x)\frac{dg}{dx}}{g^2(x)}$

(7) $\frac{d}{dx}(f \circ g)(x) = \frac{d}{dx}f(g(x))\frac{dg}{dx}(x)$ *(Chain Rule).*

Example 3.2.1 Find the derivatives of the functions

(1) $\frac{d}{dx}(x^5 + 3x^2 + 1) = 5x^4 + 6x$

(2) $\frac{d}{dx}(x^3 + x + 10)(x^4 + x^2 - 20) = (3x^2 + 1)(x^4 + x^2 - 20) + (x^3 + x + 10)(4x^3 + 2x)$

(3) $\frac{d}{dx}\frac{x+1}{x^2+1} = \frac{x^2+1-(x+1)(2x)}{(x^2+1)^2} = \frac{1-2x-x^2}{(x^2+1)^2}$

(4) $\frac{d}{dx}\frac{1}{x^2+1} = \frac{-2x}{(x^2+1)^2}$

(5) $\frac{d}{dx}(x^3 + 2x)^4 = 4(x^3 + 2x)^3(3x^2 + 2)$

Example 3.2.2 Where does the graph of $f(x) = x^4 - 2x^2 + 2$ have horizontal tangent? The curve $f(x)$ has horizontal tangent if $f'(x) = 0$. So, $f'(x) = 4x^3 - 4x = 0$, then $4x(x^2 - 1 = 4x(x-1)(x+1) = 0$. We find that $f'(x) = 0$ if $x = 0, 1, -1$.

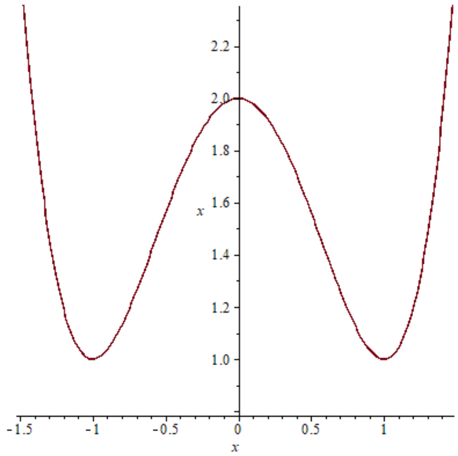


Figure 3.3: Graph of $f(x) = x^4 - 2x^2 + 2$

3.3 Derivatives of Trigonometric functions

(1) $\frac{d}{dx}(\sin x) = \cos x$.

(2) $\frac{d}{dx}(\cos x) = -\sin x$.

(3) $\frac{d}{dx}(\tan x) = \sec^2 x$.

(4) $\frac{d}{dx}(\sec x) = \sec x \tan x$.

(5) $\frac{d}{dx}(\csc x) = -\csc x \cot x$.

(6) $\frac{d}{dx}(\cot x) = -\csc^2 x$.

To prove (1), we need the following

$$\sin(x+h) = \sin x \cos h + \cos x \sin h$$

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}, \quad \text{let} \quad \theta = \frac{h}{2}, \quad \text{then} \quad \sin^2\left(\frac{h}{2}\right) = \frac{1-\cos h}{2}$$

So, we get

$$1 - \cos h = 2\sin^2\left(\frac{h}{2}\right) \quad \Rightarrow \quad \cos h - 1 = -2\sin^2\left(\frac{h}{2}\right)$$

$$\lim_{h\to 0}\frac{\cos h-1}{h}=\lim_{h\to 0}-2\frac{\sin^2\left(\frac{h}{2}\right)}{h}=\lim_{h\to 0}-2\frac{\sin\left(\frac{h}{2}\right)}{h}\sin\left(\frac{h}{2}\right)=-1.0=0$$

We prove (1)

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x(\cos h-1)+\cos x\sin h}{h}\\
&= \lim_{h\to 0}\frac{\sin x(\cos h-1)}{h}+\lim_{h\to 0}\frac{\cos x\sin h}{h}\\
&= \sin x\lim_{h\to 0}\frac{\cos h-1}{h}+\cos x\lim_{h\to 0}\frac{\sin h}{h}\\
&= \sin x.0+\cos x.1\\
&= \cos x
\end{aligned}$$

Similarly, we prove (2)

$$\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}\\
&= \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}\\
&= \lim_{h\to 0}\frac{\cos x(\cos h-1)-\sin x\sin h}{h}\\
&= \lim_{h\to 0}\frac{\cos x(\cos h-1)}{h}-\lim_{h\to 0}\frac{\sin x\sin h}{h}\\
&= \cos x\lim_{h\to 0}\frac{\cos h-1}{h}-\sin x\lim_{h\to 0}\frac{\sin h}{h}\\
&= \sin x.0-\sin x.1\\
&= -\sin x
\end{aligned}$$

Derivative of other trigonometric functions. The derivative of $y = \tan x$

$$\begin{aligned}
\frac{d}{dx}\tan x &= \frac{d}{dx}\frac{\sin x}{\cos x} \\
&= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x
\end{aligned}$$

The derivative $y = \cot x$

$$\begin{aligned}
\frac{d}{dx}\cot x &= \frac{d}{dx}\frac{\cos x}{\sin x} \\
&= \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} \\
&= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} \\
&= -\frac{1}{\sin^2 x} \\
&= -\csc^2 x
\end{aligned}$$

The derivative of $y = \sec x$

$$\begin{aligned}
\frac{d}{dx}\sec x &= \frac{d}{dx}\frac{1}{\cos x} \\
&= \frac{-(-\sin x)}{\cos^2 x} \\
&= \frac{1}{\cos x}\frac{\sin x}{\cos x} \\
&= \sec x \tan x
\end{aligned}$$

Finally, the derivative of $y = \csc x$

$$\begin{aligned} \frac{d}{dx}\csc x &= \frac{d}{dx}\frac{1}{\sin x} \\ &= \frac{-\cos x}{\sin^2 x} \\ &= -\frac{1}{\sin x}\frac{\cos x}{\sin x} \\ &= -\csc x \cot x \end{aligned}$$

Example 3.3.1 Find the derivatives of the following functions:

1. $\frac{d}{dx}\frac{1}{\sin x+\cos x} = -\frac{\cos x-\sin x}{(\sin x+\cos x)^2} = \frac{\sin x-\cos x}{(\sin x+\cos x)^2}$
2. $\frac{d}{dt}\frac{\tan t}{1+\sec t} = \frac{(1+\sec t)\sec^2 t-\tan t(\sec t\tan t)}{(1+\sec t)^2}$
3. $\frac{d}{dx}\tan(\sqrt{x}) = (\sec^2\sqrt{x})\frac{1}{2\sqrt{x}}$.
4. $\frac{d}{d\theta}\cos(\sin\theta) = -\sin(\sin\theta)\cos\theta$
5. $\frac{d}{ds}\cot\left(\frac{1}{s}\right) = -\csc^2\left(\frac{1}{s}\right)\left(\frac{-1}{s^2}\right) = \csc^2\left(\frac{1}{s}\right)\left(\frac{1}{s^2}\right)$
6. $\frac{d}{dx}(\sec x\tan x) = \sec^3 x + \sec x\tan^2 x$.

Example 3.3.2 Find the equation of the tangent line to the curve $f(x) = \sec x\tan x$ at $x = \frac{\pi}{4}$.

From the above example, the slope of the tangent line is $f'(\frac{\pi}{4}) = \sec^3(\pi/4)+\sec(\pi/4)\tan(\pi/4) = 3\sqrt{2}$ and $f(\frac{\pi}{4}) = \sqrt{2}$, so the line passes through the point $\left(\frac{\pi}{4}, \sqrt{2}\right)$. Then, the equation of the tangent line to the curve $f(x)$ at the point $\left(\frac{\pi}{4}, \sqrt{2}\right)$ is

$$y - \sqrt{2} = 3\sqrt{2}(x - \frac{\pi}{4})$$

We can find higher order derivatives, for example, if $y = x^3 + x^2$ then $y' = 3x^2 + 2x$, $y'' = 6x + 2$, $y''' = 6$.

3.4 Implicit differentiation

In this section, we consider equations that define relation between x and y. We will learn how to find $\frac{dy}{dx}$ using implicit differentiation. Let us consider some examples:

Example 3.4.1 The equation $x^2 + y^2 = 1$ defines the unit circle (the circle with center $(0, 0)$ and radius one). To find y', we differentiate both sides with respect to x to get $2x + 2yy' = 0$, from which we find that $y' = -x/y$.

We can differentiate again to find the second order derivative y''.

$$y'' = \frac{d^2y}{dx^2} = \frac{-y + xy'}{y^2} = \frac{-y + x(\frac{-x}{y})}{y^2} = -\frac{x^2 + y^2}{y^3} = \frac{-1}{y^3}$$

Example 3.4.2 Consider the implicit equation $xy = \cot(xy)$. Differentiate both sides with respect to x. Then

$$y + x\frac{dy}{dx} = -\csc^2(xy)(y + x\frac{dy}{dx}) \;\Rightarrow\; (x + \csc^2(xy))\frac{dy}{dx} = -y - y\csc^2(xy)$$

From which we find that

$$\frac{dy}{dx} = \frac{-y - y\csc^2(xy)}{x + x\csc^2(xy)} = \frac{-y\cancel{(1 + \csc^2(xy))}}{x\cancel{(1 + \csc^2(xy))}} = -\frac{y}{x}$$

Example 3.4.3 The point $(1, 1)$ lies on the curve $x^3 + y^3 - 2xy = 0$. Then find the tangent and normal to the curve there. Differentiating implicitly, we get

$$3x^2 + 3y^2\frac{dy}{dx} - 2y - 2x\frac{dy}{dx} = 0 \;\Rightarrow\; 3x^2 - 2y + (3y^2 - 2x)\frac{dy}{dx} = 0$$

from which we get

$$\frac{dy}{dx} = -\frac{3x^2 - 2y}{3y^2 - 2x}$$

The slope of the tangent line at $(1,1)$ equals -1 and the slope of the normal line equals 1. So, the equation of the tangent line and normal line are

$$\text{Tangent line } \; y-1=-(x-1), \quad \text{normal line } \; y-1=x-1$$

So, the equation of the tangent line is $y=2-x$ and the equation of the normal line is $y=x$.

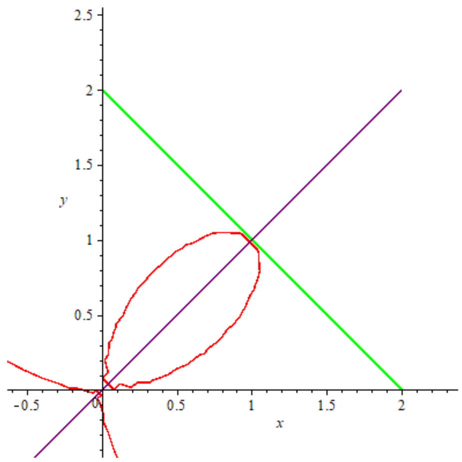


Figure 3.4: Plot of $x^3+y^3-2xy=0$ and its tangent and normal lines at $(1,1)$

Example 3.4.4 Find the two points where the curve $x^2+xy+y^2=7$ crosses the $x-$axis and show that the tangents to the curve at these points are parallel. The curve crosses the $x-$axis when $y=0$, so we get $x^2=7$ and $x=\pm\sqrt{7}$. Then, the curve crosses the $x-$axis at $(\pm\sqrt{7},0)$. Now, we find y'.

$$2x+y+x\frac{dy}{dx}+2y\frac{dy}{dx}=0 \Rightarrow (2x+y)+(x+2y)\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{-2x-y}{x+2y}$$

when $y=0$, we get

$$\frac{dy}{dx}=\frac{-2x}{x}=-2$$

3.5 Linearization and Differentials

Sometimes, we need to approximate a given nonlinear function with a linear function at some point near $(a, f(a))$. The best linear function that approximates $f(x)$ near $x = a$, provided that f is differentiable at $x = a$, is its tangent line whose equation is given by

$$L(x) = f(a) + f'(a)(x - a)$$

$L(x)$ is called the **linearization of** $f(x)$ **at** $x = a$ and the approximation $f(x) \approx L(x)$ is called the **standard linear approximation of** f **at** a.

Example 3.5.1 Find the linearization of the function $f(x) = \sqrt{1+x}$ at $x = 0$. We find that $f(0) = 1$ and $f'(x) = \frac{1}{2}(1+x)^{-1/2}$, so $f'(0) = \frac{1}{2}$. The linearization of f at $x = 0$ is $L(x) = 1 + \frac{1}{2}x$.

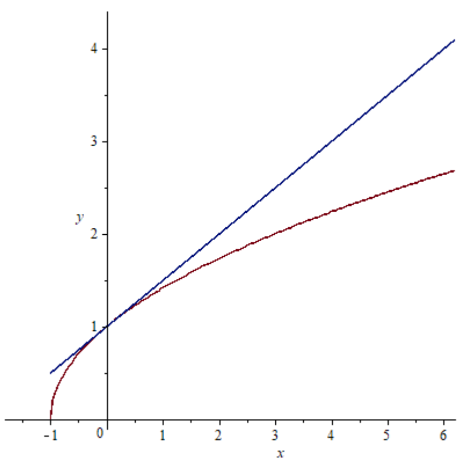


Figure 3.5: Plot of $f(x) = \sqrt{1+x}$ and its linearization $L(x) = 1 + \frac{x}{2}$

We can use the linearization to approximate the values of f near $x = 0$. Of course, the closer is x to 0, the better is the approximation.

x	Approximation	True value	|True value-Approx.|
0.2	$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.1$	1.095445	$< 10^{-2}$
0.05	$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$< 10^{-3}$
0.005	$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497	$< 10^{-5}$

Example 3.5.2 Find the linearization of the function $f(x) = \sqrt{1+x}$ at $x = 3$. Note that $f(3) = 2$, $f'(x) = \frac{1}{2}(1+x)^{-1/2}$, so $f'(3) = \frac{1}{4}$. The linearization of $f(x)$ at $x = 3$ is given by

$$L(x) = 2 + \frac{1}{4}(x-3)$$

We plot the graph of $f(x)$ with its linearizations at $x = 0$ and $x = 3$.

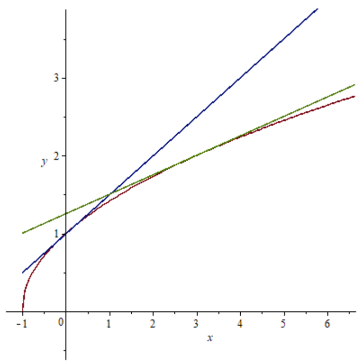


Figure 3.6: The graph of $f(x)$ with its linearizations

Example 3.5.3 Find the linearization of the function $f(x) = \sec x$ at $x = \frac{\pi}{4}$. We need to find $f(\frac{\pi}{4})$ and $f'(\frac{\pi}{4})$. Now, $f'(x) = \sec x \tan x$, so $f'(\frac{\pi}{4}) = \sqrt{2}$ and $f(\frac{\pi}{4}) = \sqrt{2}$. Then the linearization is

$$L(x) = \sqrt{2} + \sqrt{2}(x - \frac{\pi}{4})$$

Now, suppose that we move from a point $x = a$ to a nearby point $a + dx$. The change in f is

$$\Delta f = f(a + dx) - f(a)$$

while the change in L is

$$\begin{aligned}\Delta L &= L(a+dx) - L(a)\\ &= \cancel{f(a)} + f'(a)(\cancel{a} + dx - \cancel{a}) - \cancel{f(a)}\\ &= f'(a)dx\end{aligned}$$

Now, near $x = a$, we have

$$f \approx L \quad \text{then} \quad \Delta f \approx \Delta L = f'(a)dx$$

Therefore, $f'(a)dx$ gives an approximation for Δf. The quantity $f'(a)dx$ is called the **differential of f at $x = a$**. So, we get

$$\Delta f \approx df$$

Example 3.5.4 Find the differentials of the following functions

(1) $f(x) = \tan^2 x$, then $df(x) = 2\tan x \sec^2 x \; dx$

(2) $g(x) = \frac{1}{x}$ then $df(x) = -\frac{dx}{x^2}$

Example 3.5.5 The radius r of a circle increases from 10 to 10.1 m. Use dA to estimate the increase in the circle's area A. Estimate the area of the enlarged circle and compare your estimate to the true area found by direct calculations.

Solution: The area of the circle is $A(r) = \pi r^2$. Then $dA = 2\pi r dr$. The estimated increase in the area of the circle is

$$dA = 2\pi(10)0.1 = 2\pi$$

The exact change in the area of the circle is

$$\Delta A = A(10.1) - A(10) = 102.01\pi - 100\pi = 2.02\pi$$

The estimate area of the enlarged circle is

$$A(10.1) \approx A(10) + dA = 100\pi + 2\pi = 102\pi$$

The exact value of the area of the enlarged circle is $A(10.1) = \pi(10.1)^2 = 102.01\pi$. The error in this estimation is $|102.01\pi - 102\pi| = 0.01\pi$.

3.6 Exercises

1. Find the derivatives of the following functions:
   (a) $f(s) = \frac{\sqrt{s}-1}{\sqrt{s}+1}$
   (b) $f(x) = (\frac{1}{x} - x)(x^2+1)$
   (c) $g(x) = \sec(2x+1)\cot(x^2)$
   (d) $s(t) = \frac{1+\csc t}{1-\csc t}$
   (e) $f(x) = x^3 \sin x \cos x$.
   (f) $x^{1/2} + y^{1/2} = 1$.
2. Find $\frac{dy}{dx}$ for the following:
   (i) $y = \cot^2 x$
   (ii) $x^2 + y^2 = x$.
   (iii) $y = \frac{\sin x}{1-\cos x}$.
3. Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is parallel to the $x-$axis.
4. For what values of the constant a, if any, is

$$f(x) = \begin{cases} \sin(2x) & , \quad x \le 0 \\ ax & , \quad x > 0 \end{cases}$$

   (i) continuous at $x = 0$?
   (ii) Differentiable at $x = 0$.

5. Find the normals to the curve $xy + 2x - y = 0$ that are parallel to the line $2x + y = 0$.

6. Find the linearization of the following functions at the given points

   (a) $f(x) = \tan x$, $x = \pi/4$.

   (b) $g(x) = \frac{1}{x}$, $x = 1$.

   (c) $h(x) = \frac{x^2}{x^2+1}$, $x = 0$.

   (d) $f(x) = 1 + \cos\theta$, $\theta = \frac{\pi}{3}$.

7. The radius of a circle is increased from 2 to 2.02 m.

   (a) Estimate the resulting change in area.

   (b) Express the estimate as a percentage of the circle's original area.

Chapter 4

Applications of derivatives

In this chapter, we show how can we use derivatives to find the periods in which a given function $f(x)$ is increasing or decreasing and the periods in which f is concave up or concave down. Moreover, we use derivatives to find the extreme values of $f(x)$.

4.1 Increasing and decreasing functions

Definition 4.1.1 *Let $f(x)$ be a function defined on an interval I. Then,*

(a) f is increasing on I if whenever $x_2 > x_1$ then $f(x_2) > f(x_1)$, for all x_1, x_2 in I.

(b) f is decreasing on I if whenever $x_2 > x_1$ then $f(x_2) < f(x_1)$, for all x_1, x_2 in I.

For example, the functions $x, x^3, \sqrt{x}$ are increasing functions, while the functions $1-x, -x^3$ and $\frac{1}{x}, x > 0$ are all decreasing. In general, it may be not easy to find the intervals over a given function is increasing or decreasing. We use the first derivative to find these intervals as in the following theorem

Theorem 4.1.1 *Suppose that f is continuous on $[a,b]$ and differentiable on (a,b) then*

(a) If $f'(x) > 0$, for all $x \in (a,b)$ then f is increasing on $[a,b]$.

(b) If $f'(x) < 0$, for all $x \in (a,b)$, then f is decreasing on $[a,b]$.

Example 4.1.1 Let $f(x) = x^3 - 12x - 5$. Then

$$f'(x) = 3x^2 - 12 = 3(x-2)(x+2)$$

Note that $f'(x) > 0$ for all $x \in (-\infty,-2) \cup (2,\infty)$ and $f'(x) < 0$ for all $x \in (-2,2)$. So, f is increasing on $(-\infty,-2] \cup [2,\infty)$ and decreasing on $[-2,2]$.

Example 4.1.2 Let $g(x) = x^3 + x^2 - x + 1$ then

$$g'(x) = 3x^2 + 2x - 1 = (3x-1)(x+1)$$

Then, $g'(x) > 0$ for all $x \in (-\infty,-1) \cup (\frac{1}{3},\infty)$ and $g'(x) < 0$ for all $x \in (-1,\frac{1}{3})$. So, g is increasing on $(-\infty,-1] \cup [\frac{1}{3},\infty)$ and is decreasing on $[-1,\frac{1}{3}]$.

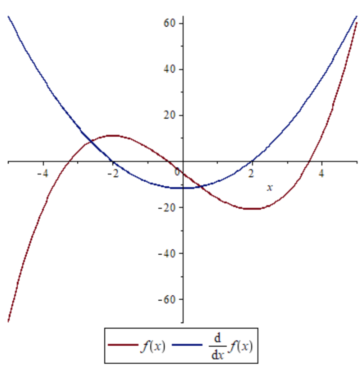


Figure 4.1: Limit of a function

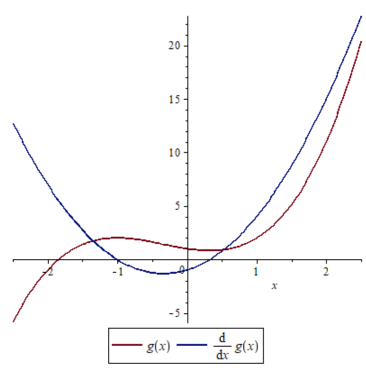


Figure 4.2: Example of limits

4.2 Extreme values of functions

Definition 4.2.1 *Let f be a function with domain D. Then,*

(a) f has an ***absolute maximum*** *value on D at a point c if $f(x) \leq f(c)$, for all $x \in D$.*

(b) f has an ***absolute minimum*** *value on D at a point c if $f(x) \geq f(c)$, for all $x \in D$.*

$f(c)$ is called local maximum (resp. local minimum) if the inequality in (a) (resp. (b)) holds in a small interval around $x = c$.

Example 4.2.1 The function $f(x) = x^3$, $D = [-1, 1]$ has absolute minimum value $f(-1) = -1$ and absolute maximum value $f(1) = 1$. Similarly, the function $f(x) = x^2$ on $[-1, 1]$ has absolute maximum at $x = \pm 1$ and absolute minimum at $x = 0$. But if we consider the functions x^2 and x^3 over the open interval $(-1, 1)$ then x^3 has neither maximum nor minimum on $(-1, 1)$ and x^2 has absolute minimum at $x = 0$.

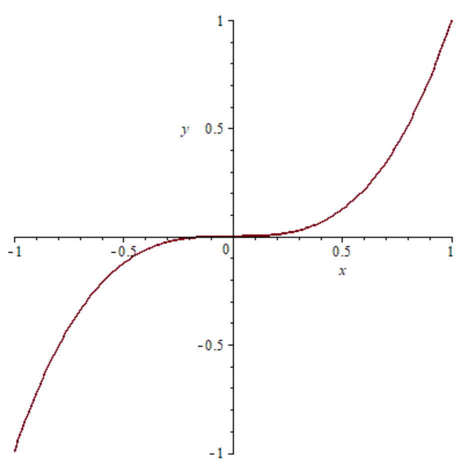


Figure 4.3: The graph of $f(x) = x^3$ on $[-1, 1]$

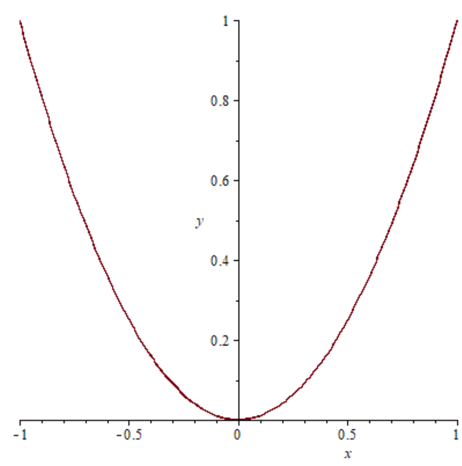


Figure 4.4: The graph of $f(x) = x^2$ on $[-1, 1]$

Theorem 4.2.1 *If f is continuous on a closed interval $[a, b]$ then f has both an absolute maximum value and an absolute minimum value.*

To find the extreme values of a function f on a closed interval, we look for these values at the endpoints of the interval and at the interior points where $f' = 0$ or undefined (**critical points**).

Definition 4.2.2 *An interior point where f' equals zero or undefined is called a critical point of f.*

Example 4.2.2 Let $f(x) = x\sqrt{1-x^2}$. The domain of this function is $D = [-1, 1]$ and f is differentiable on $(-1, 1)$ with derivative

$$f'(x) = \sqrt{1-x^2} + x\frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}}$$

Then, $f'(x) = 0$ when $1 - 2x^2 = 0$ and f has two critical points $x = \pm\frac{1}{\sqrt{2}}$.

Example 4.2.3 Let $f(x) = x^{2/3}$, $D = [-1, 8]$. The derivative of f is $f'(x) = \frac{2}{3x^{1/3}}$. Then $f'(0)$ is undefined. To find the extreme values of f, we evaluate f at the endpoints $x = -1, x = 8$ and at the critical point $x = 0$. Since $f(-1) = 1, f(0) = 0, f(8) = 4$, then $f(0) = 0$ is an absolute minimum and $f(8) = 4$ is an absolute maximum.

Theorem 4.2.2 *If f is differentiable and has an extreme value at an interior point c then $f'(c) = 0$.*

If $f'(c) = 0$, this does not mean that f has an extreme value (maximum or minimum) at $x = c$. For example, $x = 0$ is a critical point of $f(x) = x^3$ but $f(0)$ is neither maximum nor minimum for $y = x^3$.

To classify the critical as maximum or minimum, we can use either the first derivative test or the second derivative test which we state now.

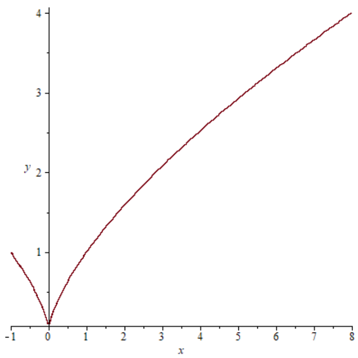


Figure 4.5: Graph of $f(x) = x^{2/3}$

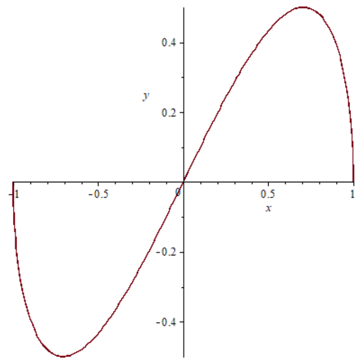


Figure 4.6: Graph of $f(x) = x\sqrt{1-x^2}$

Theorem 4.2.3 ***(First derivative test)*** *Suppose that f has a critical point at $x = c$ and that $f'(x)$ exists in an open interval containing $x = c$. Then*

(a) If f' changes sign from positive to negative at $x = c$ then $f(c)$ is a local maximum.

(b) If f' changes sign from negative to positive at $x = c$ then $f(c)$ is a local minimum.

(c) If f' does not change sign at $x = c$ then f does not have an extreme value at $x = c$.

Example 4.2.4 Consider the function $f(x) = x\sqrt{1-x^2}$ from example (4.2.2) whose derivative is

$$f'(x) = \frac{1-2x^2}{\sqrt{1-x^2}}$$

f has two critical point $x = \pm\frac{1}{\sqrt{2}}$, the sign of f' is

$$-----\frac{-1}{\sqrt{2}}+++++\frac{1}{\sqrt{2}}-----$$

So, f has a local minimum at $x = -\frac{1}{\sqrt{2}}$ and local maximum at $x = \frac{1}{\sqrt{2}}$. Its maximum value is $f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$ and its minimum value is $f\left(\frac{-1}{\sqrt{2}}\right) = -\frac{1}{2}$. In fact, as it is clear from figure (4.6) these extreme values are absolute.

Theorem 4.2.4 ***(Second derivative test)*** *Suppose that $f'(c) = 0$ and that f'' is continuous in an open interval containing c. Then*

(a) If $f''(c) < 0$ then $f(c)$ is a local maximum.

(b) If $f''(c) > 0$ then $f(c)$ is a local minimum.

(c) If $f''(c) = 0$ then the test fails.

If $f''(x) \geq 0$ for all x in an interval I then f is concave up on I. If $f''(x) \leq 0$ for all x in an interval I then f is concave down on I.

Definition 4.2.3 *A point where f has tangent line and changes concavity is called* ***an inflection point*** *of f.*

Example 4.2.5 Find the intervals at which the function

$$f(x) = x^4 - 4x^3 + 10$$

is increasing, decreasing, concave up and concave down. Then, find the extreme values of f.
Solution: The first and second derivatives of f are given by

$$f'(x) = 4x^2(x-3) \quad \text{and} \quad f''(x) = 12x(x-2)$$

We find that $f'(x) = 0$ at $x = 0$ and $x = 3$, $f''(x) = 0$ at $x = 0$ and $x = 2$, so f has two critical points $x = 0$ and $x = 3$. The signs of f' and f'' are found to be as

$$f' \quad - - - - - - - - 0 - - - - - - - - 3 + + + + + + +$$

$$f'' \quad + + + + + + + 0 - - - - - 2 + + + + + + +$$

Hence, $f'(x) < 0$ for all $x \in (-\infty, 0) \cup (0, 3)$ and $f'(x) > 0$ for all $x \in (3, \infty)$. We conclude that f is decreasing on $(-\infty, 3]$ and f is increasing on $[3, \infty)$. It follows that $f(3) = -17$ is an absolute minimum.

Moreover, $f''(x) > 0$ for all $x \in (-\infty, 0) \cup (2, \infty)$ and $f''(x) < 0$ for all $x \in (0, 2)$. We conclude that f is concave up on $(-\infty, 0] \cup [2, \infty)$ and f is concave down on $[0, 2]$. Finally, f has inflection points at $(0, 10)$ and $(2, -6)$.

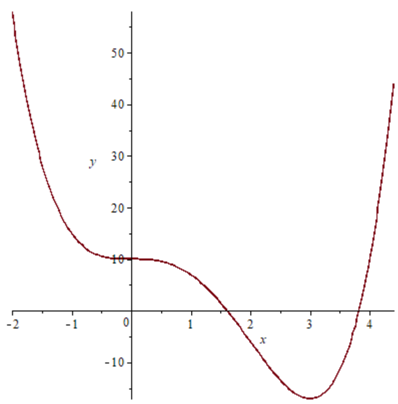


Figure 4.7: Graph of $y = x^4 - 4x^3 + 10$

Example 4.2.6 Consider the function

$$f(x) = \frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$$

Then,

$$f'(x) = \frac{(x+1)2x - x^2}{(x+1)^2} = \frac{x^2+2x}{(x+1)^2} = \frac{x(x+2)}{(x+1)^2}$$

and

$$\begin{aligned} f''(x) &= \frac{(x+1)^2(2x+2) - (x^2+2x)(2)(x+1)}{(x+1)^4} \\ &= \frac{2(x+1)^2 - 2(x^2+2x)}{(x+1)^3} \\ &= \frac{2x^2+4x+2-2x^2-4x}{(x+1)^3} \\ &= \frac{2}{(x+1)^3} \end{aligned}$$

(1) Domain of f: $(-\infty, \infty)\backslash\{-1\}$

(2) $\lim_{x\to+\infty} \frac{x^2}{x+1} = \lim_{x\to+\infty} \frac{x}{1+\frac{1}{x}} = +\infty$

(3) $\lim_{x\to-\infty} \frac{x^2}{x+1} = \lim_{x\to-\infty} \frac{x}{1+\frac{1}{x}} = -\infty$

(4) Horizontal asymptotes: None

(5) $\lim_{x\to-1^+} \frac{x^2}{x+1} = +\infty$

(6) $\lim_{x\to-1^-} \frac{x^2}{x+1} = -\infty$

(7) Vertical asymptote: $x = -1$

(8) Oblique asymptote $y = x - 1$

(9) Critical points $x = 0, -2$ since $f'(x) = 0$ at $x = 0$, $x = -2$

(10) f' $+++++(-2)---(-1)---0+++++$, so f is increasing on $(-\infty,-2]\cup[0,\infty)$ and decreasing on $[-2,-1)\cup(-1,0]$

(11) $f(-2)=-4$ is a local maximum.

(12) $f(0)=0$ is a local minimum.

(13) f'' $-----(-1)+++++$, so f is concave down on $(-\infty,-1)$ and concave up on $(-1,\infty)$

(14) Absolute maximum and absolute minimum values: None.

(15) Inflection points: None.

(16) Range of f: $(-\infty,-4]\cup[0,\infty)$

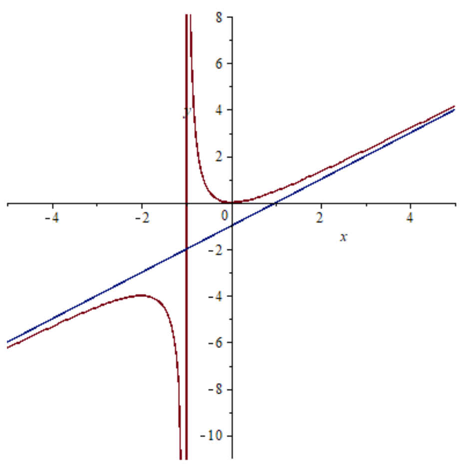


Figure 4.8: Graph of $f(x)=\frac{x^2}{x+1}$ and its asymptotes

Example 4.2.7 Consider the function

$$f(x) = \frac{x^2}{x^2 - 1}$$

Then

$$f'(x) = \frac{(x^2-1)(2x) - x^2(2x)}{(x^2-1)^2} = \frac{2x^3 - 2x - 2x^3}{(x^2-1)^2} = \frac{-2x}{(x^2-1)^2}$$

and

$$\begin{aligned} f''(x) &= \frac{(x^2-1)^2(-2) + 2x(2)(2x)(x^2-1)}{(x^2-1)^4} \\ &= \frac{-2(x^2-1) + 8x^2}{(x^2-1)^3} \\ &= \frac{6x^2+2}{(x^2-1)^3} \end{aligned}$$

(1) Domain $(-\infty, \infty)\backslash\{\pm 1\}$

(2) $\lim\limits_{x \to \pm\infty} \frac{x^2}{x^2-1} = 1$

(3) Horizontal asymptote $y = 1$

(4) $\lim\limits_{x \to 1^+} \frac{x^2}{x^2-1} = +\infty$

(5) $\lim\limits_{x \to 1^-} \frac{x^2}{x^2-1} = -\infty$

(6) $\lim\limits_{x \to -1^+} \frac{x^2}{x^2-1} = -\infty$

(7) $\lim\limits_{x \to -1^-} \frac{x^2}{x^2-1} = +\infty$

(8) Vertical asymptotes: $x = 1$ and $x = -1$

(9) Critical point $x = 0$ since $f'(0) = 0$

(10) f' $+++++(-1)+++++0-----1-----$, so f is increasing on $(-\infty,-1)\cup(-1,0]$ and f is decreasing on $[0,1)\cup(1,\infty)$

(11) $f(0)=0$ is a local maximum.

(12) Local minimum: None

(13) Absolute maximum and absolute minimum: None

(14) f'' $+++++(-1)-----1+++++$, so f is concave up on $(-\infty,-1)\cup(1,\infty)$ and concave down on $(-1,1)$.

(15) Inflection points: None.

(16) Range of f: $(-\infty,0]\cup(1,\infty)$

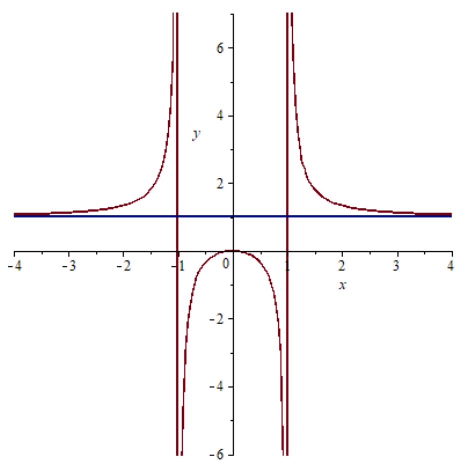


Figure 4.9: Graph of $f(x)=\frac{x^2}{x^2-1}$ and its asymptotes

Example 4.2.8 Consider the function

$$f(x) = \frac{x}{x^2+1}$$

Then

$$f'(x) = \frac{(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

$$\begin{aligned} f''(x) &= \frac{(x^2+1)^2(-2x) - (1-x^2)(2)(1+x^2)(2x)}{(x^2+1)^2} \\ &= \frac{-2x(x^2+1) - 4x(1-x^2)}{(x^2+1)^3} \\ &= \frac{2x^3-6x}{(x^2+1)^3} \\ &= \frac{2x(x^2-3)}{(x^2+1)^3} \end{aligned}$$

(1) Domain: $(-\infty, \infty)$

(2) $\lim_{x\to\pm\infty} \frac{x}{x^2+1} = 0$

(3) Horizontal asymptote: $y = 0$

(4) Vertical asymptote: None

(5) Oblique asymptote: None

(6) Critical points: $x = 1$ and $x = -1$ since $f'(\pm 1) = 0$

(7) f' $\quad - - - - - (-1) + + + + + 1 - - - - - -$, so f is increasing on $[-1, 1]$ and f is decreasing on $(-\infty, -1] \cup [1, \infty)$

(8) Local maximum $f(1) = \frac{1}{2}$

(9) Local minimum $f(-1) = -\frac{1}{2}$

(10) Absolute maximum $f(1) = \frac{1}{2}$

(11) Absolute minimum $f(-1) = -\frac{1}{2}$

(12) f'' $\quad - - - - - (-\sqrt{3}) + + + + + 0 - - - - - \sqrt{3} + + + + +$, so f is concave up on $[-\sqrt{3}, 0] \cup [\sqrt{3}, \infty)$ and f is concave down on $(-\infty, -\sqrt{3}] \cup [0, \sqrt{3}]$

(13) Inflection points $(-\sqrt{3}, \frac{-\sqrt{3}}{4}), (0, 0), (\sqrt{3}, \frac{\sqrt{3}}{4})$

(14) Range of f: $[-\frac{1}{2}, \frac{1}{2}]$

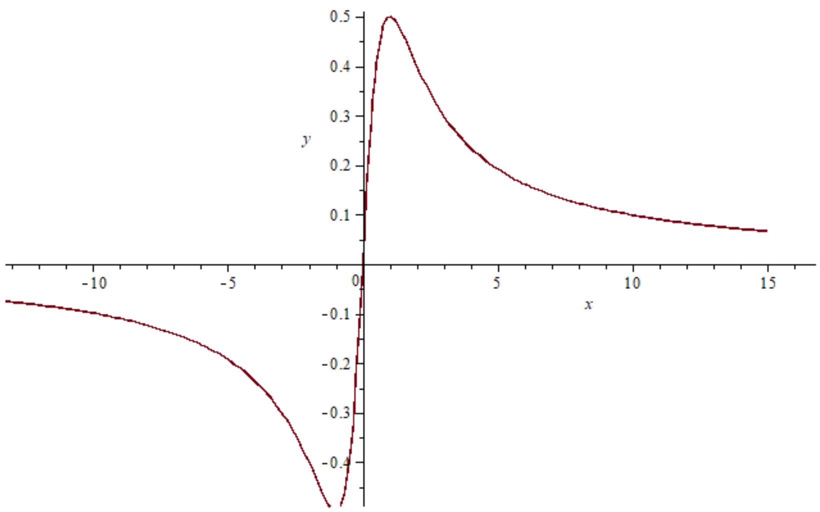


Figure 4.10: Graph of $f(x) = \frac{x}{x^2+1}$

4.3 The Mean Value Theorem

Theorem 4.3.1 ***Rolle's Theorem*** *If $y = f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on (a, b) and $f(a) = f(b)$, then there is at least one point c in (a, b) such that $f'(c) = 0$.*

Theorem 4.3.2 ***The Mean Values Theorem*** *If $y = f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on (a, b), then there is at least one point c in (a, b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

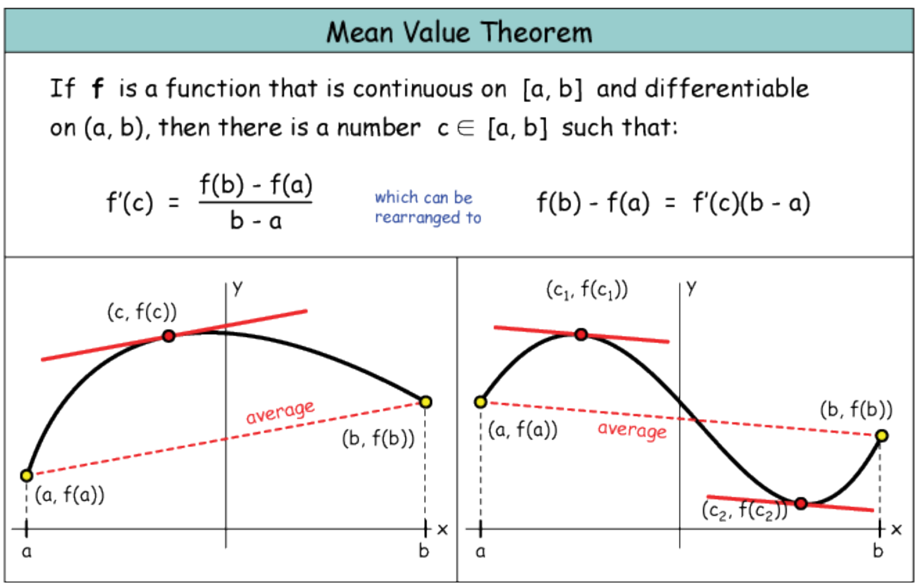


Figure 4.11: Graph of $f(x) = \frac{x^2}{x^2-1}$ and its asymptotes

The mean value theorem means that, at some point c in the interval $[a, b]$, the slope of the tangent line at $(c, f(c))$ equals the slope of the secant line through the points $(a, f(a))$ and $(b, f(b))$.

Example 4.3.1 Let $f(x) = x^2$, $x \in [1, 4]$. Find the point c in the conclusion of the mean value theorem. Note that f is continuous on

$[1, 4]$ and differentiab le on $(1, 4)$. Then,

$$\frac{f(4) - f(1)}{4 - 1} = \frac{16 - 1}{4 - 1} = \frac{15}{3} = 5, \quad f'(c) = 2c \quad \Rightarrow 5 = 2c \Rightarrow \quad c = \frac{5}{2}$$

4.4 Exercises

1. Find the intervals in which the following functions are increasing, decreasing, concave up and concave down. Then, find the extreme values and inflection points and sketch their graphs:
   (a) $y = 1 - (x + 1)^3$
   (b) $y = \frac{x^2+1}{x}$
   (c) $y = x^4 - 2x^2$
   (d) $y = \frac{x^2-3}{x-2}$
   (e) $y = \sqrt[3]{x^3 + 1}$
   (f) $y = \frac{x}{x^2-1}$
   (g) $y = x\sqrt{8 - x^2}$
2. Find the value of c in the conclusion of the mean value theorem for the function $f(x) = \sqrt{x}$ on the interval $[a, b]$, $a > 0$.
3. For what values of a, m and b does the function

$$f(x) = \begin{cases} 3 & , \quad x = 0 \\ -x^2 + 3x + a & , \quad 0 < x < 1 \\ mx + b & , \quad 1 \leq x \leq 2 \end{cases}$$

   satisfy the hypotheses of the mean value theorem on the interval $[0, 2]$.

Chapter 5

Integration

5.1 Antiderivative and integration

Definition 5.1.1 *A function F is called an* ***antiderivative*** *of a function f on an interval I if $F'(x) = f(x)$, for all x in I. The set of all antiderivatives of f is called the* ***indefinite integral*** *of f and is denoted by $\int f(x)dx$.*

Example 5.1.1 An antiderivative of the function $f(x) = 2x$ is $F(x) = x^2$ since $F'(x) = 2x = f(x)$. All antiderivatives of $f(x) = x^2$ are given by $F(x) = x^2 + C$, for any constant C.

Example 5.1.2 In this example, we give the indefinite integrals of some important functions

(a) $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$

(b) $\int \sin x dx = -\cos x + C$

(c) $\int \cos x dx = \sin x + C$

(d) $\int \sec^2 x dx = \tan x + C$

(e) $\int \sec x \tan x dx = \sec x + C$

(f) $\int \csc x \cot x dx = -\csc x + C$

(g) $\int \csc^2 x dx = -\cot x + C$

Example 5.1.3 Consider the following examples:

(a) $\int (x^{-2} - x^2 + 1)dx = -\frac{1}{x} - \frac{1}{3}x^3 + x + C$

(b) $\int \cos^2 \theta d\theta = \int \frac{1+\cos(2\theta)}{2} d\theta = \frac{1}{2}\int (1 + \cos(2\theta))d\theta = \frac{1}{2}(\theta + \frac{\sin(2\theta)}{2}) + C$

(c) $\int \sin^2 x \, dx = \int \frac{1-\cos(2x)}{2} dx = \frac{1}{2}\int (1 - \cos(2x))dx = \frac{1}{2}(x - \frac{\sin(2x)}{2}) + C$

(d) $\int \cot^2 x \, dx = \int (\csc^2 x - 1)dx = -\cot x - x + C$

5.2 Definite integrals and areas

Sometimes, we evaluate integrals on given intervals. Such integrals are called definite integrals and take the form

$$\int_a^b f(x)dx$$

We can solve definite integrals using the fundamental theorem of calculus:

Theorem 5.2.1 ***Fundamental Theorem of Calculus***

(I) Suppose that f is continuous on $[a, b]$ and F is an is an antiderivative of f on $[a, b]$ then

$$\int_a^b f(x)dx = F(b) - F(a)$$

(II) Suppose that f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)dt$ then F is continuous on $[a, b]$ and differentiable on (a, b) and $F'(x) = f(x)$.

If $f(x) \geq 0$ is an integrable function on $[a, b]$ then $\int_a^b f(x)dx$ is the area enclosed between the curve $f(x)$ and the $x-$axis.

Example 5.2.1 Find the derivatives of the following functions

(a) $\frac{d}{dx}\int_0^x \sin t dt = \sin x.$

(b) $\frac{d}{dx}\int_1^{x^2} \frac{dt}{1+t^2} = \frac{2x}{1+x^4}$

(c) $\frac{d}{dx}\int_{\sin x}^1 \frac{dt}{t} = \frac{d}{dx}\left(-\int_1^{\sin x}\frac{dt}{t}\right) = -\frac{\cos x}{\sin x} = -\cot x$

(d) $\frac{d}{dx}\int_{x^2}^{x^3} \sin t \; dt = \sin(x^3)(3x^2) - \sin(x^2)(2x)$

Example 5.2.2 Find the area enclosed between the following curves and the $x-$axis in the given intervals

(a) $f(x) = 2x\sqrt{x^2+1}$, $x \in [0, 1]$. The area is given by the following integral

$$A = \int_0^1 2x\sqrt{x^2+1}dx$$

using substitution $u = x^2 + 1$, $du = 2xdx$. The integral can be written as

$$A = \int_1^2 u^{1/2}du = \frac{2}{3}u^{3/2}\bigg|_1^2 = \frac{2}{3}(2\sqrt{2}-1)$$

We can find the area enclosed between two functions $f(x)$ and $g(x)$ in some interval $[a, b]$ where $f(x) \geq g(x)$, using the formula

$$A = \int_a^b (f(x) - g(x))dx$$

Sometimes, the functions are expressed in terms of y in some interval $[c, d]$, so the area in this case is

$$A = \int_c^d (f(y) - g(y))dy$$

The next examples explain both cases.

Example 5.2.3 Find the area enclosed between the curves $f(x) = 2 - x^2$ and $y = -x$.

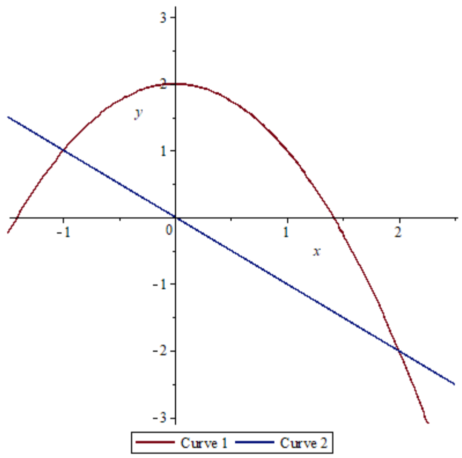


Figure 5.1: Plot of $f(x) = 2 - x^2$, $g(x) = -x$

Solution We first find the points at which the two curves intersect by equating the functions

$$-x = 2 - x^2 \quad \text{which is equivalent to} \quad x^2 - x - 2 = 0$$

The last equation can be factorized as $(x+1)(x-2) = 0$. Thus, the two curves intersect at $x = -1$ and $x = 2$. So, the area is given by

$$\begin{aligned} A &= \int_{-1}^{2} (2 - x^2 + x)dx \\ &= \left(2x - \frac{x^3}{3} + \frac{x^2}{2}\right|_{-1}^{2} \\ &= 4 - \frac{8}{3} + 2 + 2 - \frac{1}{3} - \frac{1}{2} \\ &= \frac{9}{2} \end{aligned}$$

Example 5.2.4 Find the area enclosed between the curves $y = \sqrt{x}$, the $x-$axis and the line $y = x - 2$. It is easier to write x as a function of y and to integrate with respect to y. In this case, we have $x = y^2$ and $x = y + 2$. The two curves intersect at the point $y = 2$. The area is given by the integral

$$\begin{aligned} A &= \int_0^2 (y + 2 - y^2) dy \\ &= \left(\frac{y^2}{2} + 2y - \frac{y^3}{3} \right|_0^2 \\ &= 2 + 4 - \frac{8}{3} \\ &= \frac{10}{3} \end{aligned}$$

integrating with respect to x,

$$A = \int_0^2 \sqrt{x} dx + \int_2^4 (\sqrt{x} - (x - 2)) dx = \frac{10}{3} \quad (check!!!)$$

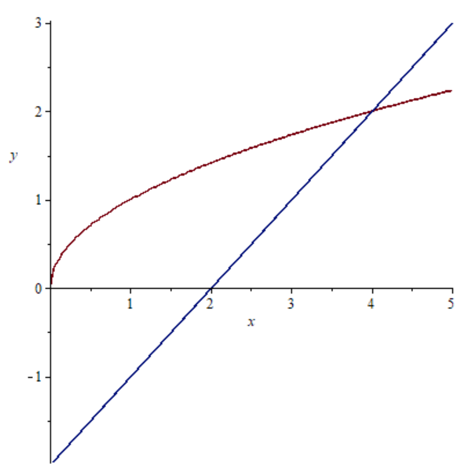


Figure 5.2: Plot of $y = \sqrt{x}$ and $y = x - 2$

5.3 Additional Examples

Example 5.3.1 Solve $\int \sqrt{\frac{x^4}{x^3-1}}dx$

$$\int \sqrt{\frac{x^4}{x^3-1}}dx = \int \frac{x^2}{\sqrt{x^3-1}}dx$$

using the substitution $u = x^3 - 1$, $du = 3x^2dx$, the integral becomes

$$\frac{1}{3}\int \frac{du}{u^{1/2}} = \frac{1}{3}\int u^{-1/2}du = \frac{2}{3}\sqrt{u} = \frac{2}{3}\sqrt{x^3-1} + C$$

Example 5.3.2 Find the area enclosed between the curve $f(x) = x^{1/3} - x$ and the $x-$axis in the interval $[-1, 8]$. Notice that $f(x) = 0$ at $x = -1, 0, 1$, and its graph lies below the $x-$axis in the intervals $[-1, 0]$, $[1, 8]$ and above the $x-$axis in the interval $[0, 1]$. So,

$$\begin{aligned}
A &= \left|\int_{-1}^{0}(x^{1/3} - x)dx\right| + \int_{0}^{1}(x^{1/3} - x)dx + \left|\int_{1}^{8}(x^{1/3} - x)dx\right| \\
&= \left|\frac{3}{4}x^{4/3} - \frac{x^2}{2}\Big|_{-1}^{0}\right| + \left(\frac{3}{4}x^{4/3} - \frac{x^2}{2}\Big|_{0}^{1}\right) + \left|\frac{3}{4}x^{4/3} - \frac{x^2}{2}\Big|_{1}^{8}\right| \\
&= \left|-\frac{3}{4} + \frac{1}{2}\right| + \left(\frac{3}{4} - \frac{1}{2}\right) + \left|12 - 32 - \frac{3}{4} + \frac{1}{2}\right| \\
&= \frac{1}{4} + \frac{1}{4} + \frac{81}{4} \\
&= \frac{83}{4}
\end{aligned}$$

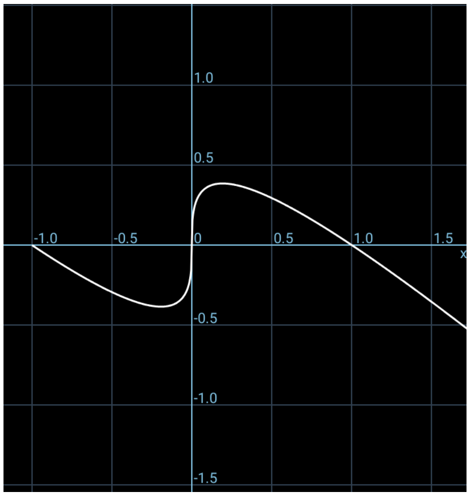


Figure 5.3: Plot of $f(x) = x^{1/3} - x$

5.4 Exercises

1. Solve the following integrals:
   (a) $\int \sin(5x)dx$
   (b) $\int \tan^2 x dx$
   (c) $\int (1 + \cot^2 \theta)d\theta$.
   (d) $\int \frac{\csc \theta d\theta}{\csc \theta - \sin \theta}$
2. Find the derivatives of the following functions
   (a) $y = \int_1^x \frac{dt}{t}$
   (b) $y = \int_0^{\sqrt{x}} \cos t dt$
   (c) $y = \int_{\tan x}^0 \frac{dt}{1+t^2}$
3. Find the linearization of $g(x) = 3 + \int_1^{x^2} \sec(t-1)dt$ at $x = -1$
4. Solve the following definite integrals
   (a) $\int_1^{\sqrt{2}} \frac{s^2+\sqrt{s}}{s^2} ds$

(b) $\int_0^{\pi/6}(\sec x + \tan x)^2 dx$

(c) $\int_0^{\pi}(\cos x + |\cos x|)dx$

5. Use substitution to solve the following integrals:

(a) $\int \frac{dx}{\sqrt{x}(1+\sqrt{x})^2}$

(b) $\int \frac{\sec z \tan z}{\sqrt{\sec z}} dz$

(c) $\int \sqrt{\frac{x-1}{x^5}} dx$

(d) $\int x^3\sqrt{x^2+1}dx$

6. Find the area enclosed between the given functions:

(a) $y = x^2 - 2x$, $y = x$

(b) $y = x^2$, $y = -x^2 + 4x$

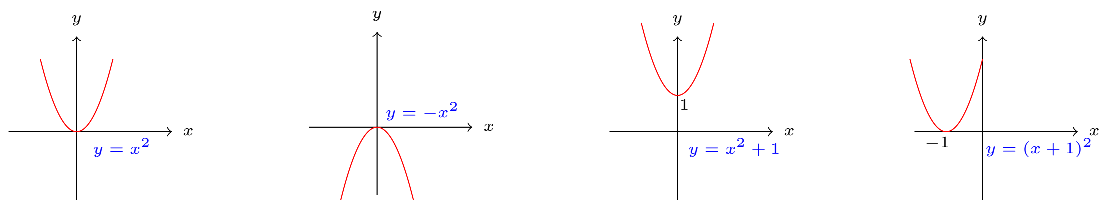
(c) $x = y^2$, $x = 3 - 2y^2$

(d) $x = y^3 - y^2$, $x = 2y$

Basics (Self Study)

- Functions: are maps in which every x value has only one image $f(x) = y$
- y-intercept: Where f crosses y-axis $\rightarrow$ Let $x = 0$, then find $y = f(0)$
- x-intercept (zero or root): Where f crosses x-axis $\rightarrow$ Let $y = 0$, then find x
- Shifting and reflections: Given a function $y = f(x)$ and a constant $c > 0$, then

  1) $y = f(x) + c$: Shift the graph of $f(x)$ c units upward.
  2) $y = f(x) - c$: Shift the graph of $f(x)$ c units downward.
  3) $y = f(x + c)$: Shift the graph of $f(x)$ c units leftward.
  4) $y = f(x - c)$: Shift the graph of $f(x)$ c units rightward.
  5) $y = -f(x)$: Reflect the graph of $f(x)$ about x-axis.
  6) $y = f(-x)$: Reflect the graph of $f(x)$ about y-axis



- Linear functions (Lines):
- General Form: $y = f(x) = mx + b$, where $m = \frac{\Delta y}{\Delta x} = y'$ is the slope of the line.
- $(y - y_0) = m(x - x_0)$: Gives the equation of the line with slope m and passes through (x_0, y_0)
- : Horizontal line: $y = c \rightarrow$ Slope $= 0$
- : Vertical line: $x = c \rightarrow$ Slope undefined
- : If L_1 and L_2 are two lines with slopes m_1 and m_2 respectively, then

  1) L_1 and L_2 are parallel if $m_1 = m_2$
  2) L_1 and L_2 are perpendicular (normal) if $m_1 = -\frac{1}{m_2}$

- Solving Equations and inequalities with absolute value:
  - $|x| = a \rightarrow x = \pm a$
  - $|x| \leq a \rightarrow -a \leq x \leq a$
  - $|x| \geq a \rightarrow x \leq -a$ or $x \geq a$

- Special Factorizations:
  - $x^2 - a^2 = (x - a)(x + a)$
  - $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$
  - $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$

- Quadratic functions (Parabolas):
- General Form: $y = f(x) = ax^2 + bx + c$; $a \neq 0$
- Vertex: is the point $(\frac{-b}{2a}, f(\frac{-b}{2a}))$
- Discriminant$= b^2 - 4ac$

  1) If discriminant > 0, then $f(x)$ has two real roots.

  2) If discriminant $= 0$, then $f(x)$ has one real root.

  3) If discriminant < 0, then $f(x)$ has no real roots.
- Quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

  If $a > 0$ then the parabola is open upward (concave up)

  If $a < 0$ then the parabola is open downward (concave down)
- Square Completion: Given $x^2 + bx + c$, (notice that $a = 1$), add $\pm(\frac{b}{2})^2$

  $\rightarrow x^2 + bx + c = x^2 + bx + (\frac{b}{2})^2 - (\frac{b}{2})^2 + c = (x - |\frac{b}{2}|)^2 - (\frac{b}{2})^2 + c$

  Ex: $x^2 - 6x + 11 = x^2 - 6x + 9 - 9 + 11 = (x - 3)^2 + 2$

- Special Quadratic Curves in y: $x = y^2$ and $x = -y^2$

  $x = y^2$: a parabola open to the right with vertex $(0, 0)$

  $x = -y^2$: a parabola open to the left with vertex $(0, 0)$

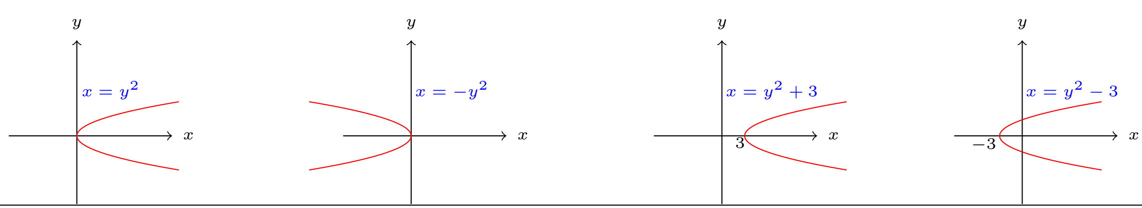
  Examples of shifts on $x = y^2$:

  1) $x = y^2 + 3$: Shift the graph of $x = y^2$ three units to the right

  2) $x = y^2 - 3$: Shift the graph of $x = y^2$ three units to the left

  3) $x = (y + 3)^2$: Shift the graph of $x = y^2$ three units downward

  4) $x = (y - 3)^2$: Shift the graph of $x = y^2$ three units upward



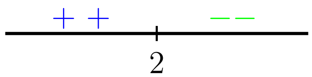
- Circles:

  $(x - a)^2 + (y - b)^2 = r^2$: a circle with center (a, b) and radius r
- Unit circle: $x^2 + y^2 = 1$: center$= (0, 0)$ and radius $= 1$

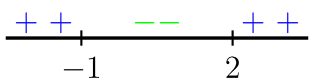
- Determine the sign of $y = f(x)$: Sometimes we need to know when y is positive (above x-axis) and when y is negative (below x-axis)

1) Polynomials: Find the zeros, if any, then substitute values

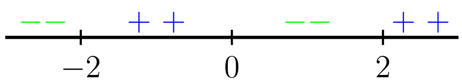
Ex: $f(x) = 4 - 2x \to 4 - 2x = 0 \to x = 2$ (take $f(0) = 4 > 0$ but $f(3) = -2 < 0$)



Ex: $f(x) = x^2 - x - 2 \to x^2 - x - 2 = 0 \to x = -1, 2$
($f(-2) = 4 > 0$, $f(0) = -2 < 0$, $f(3) = 4 > 0$)



Ex: $f(x) = x^3 - 4x \to x^3 - 4x = 0 \to x = -2, 0, 2$



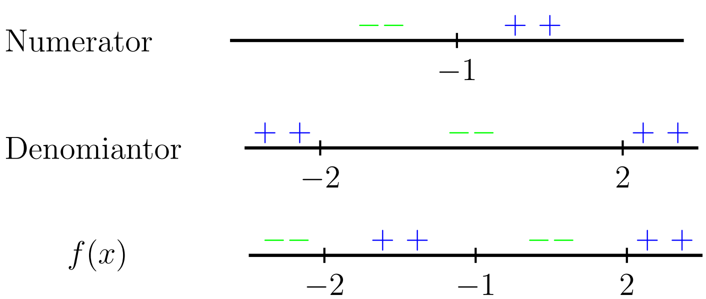
Ex: $f(x) = x^2 + 3$ has no zeros, so substitute any value $f(1) = 4 > 0$

2) Rational functions $= \frac{\text{polynomial}}{\text{polynomial}}$: Determine sign of numerator, then denominator, then divide

Ex: $f(x) = \frac{x^3+1}{x^2-4}$

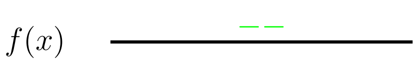
Numerator: $x^3 + 1 = 0 \to x = -1$

Denominator: $x^2 - 4 = 0 \to x = -2, 2$

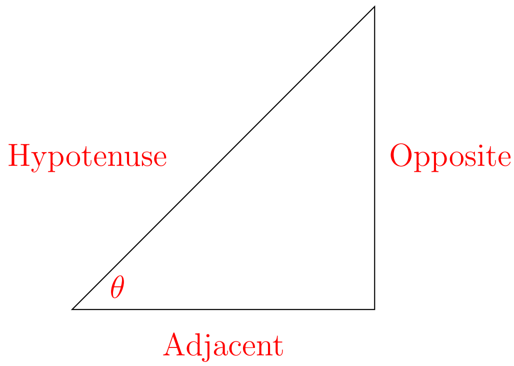


Ex: $f(x) = \frac{-2}{x^2+1}$

'The numerator is always negative and the denominator is always positive, so f is always negative.



- Trigonometric functions



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{1}{\sin\theta}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin\theta}{\cos\theta} = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$$

θ	$\sin\theta$	$\cos\theta$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0
π	0	-1
$\frac{3\pi}{2}$	-1	0
2π	0	1

- Unit Circle and trigonometric functions:

  Recall: Unit Circle: $x^2 + y^2 = 1$ and $\cos^2\theta + \sin^2\theta = 1$

  $\rightarrow$ For any point on this circle: $(x, y) = (\cos\theta, \sin\theta)$,where θ : is the angle (counterclockwise) between the positive x-axis and the line segment form origin to point (x, y)

Ex: $(\frac{\sqrt{3}}{2}, \frac{1}{2}) = (\cos(\frac{\pi}{6}), \sin(\frac{\pi}{6}))$, $(0, 1) = (\cos(\frac{\pi}{2}), \sin(\frac{\pi}{2}))$, $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = (\cos(\frac{3\pi}{4}), \sin(\frac{3\pi}{4}))$