Lectures 2 and 3: Review of limits and continuity

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0.1 Limits and continuity(2 lectures)

[1]

0.2 Limits of functions

When a function f approaches a certain limit L as x approaches x_0, we write

$$\lim_{x\to x_0} f(x) = L$$

This limit means that *the function gets arbitrarily close to* L *when* x *is sufficiently close to* x_0. Notice that x_0 or L or both of them can be $+\infty$ or $-\infty$. The function f may or may not be defined at x_0. As you know,

$$\lim_{x\to x_0} f(x) = L \quad \text{if and only if} \lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = L$$

Example 0.2.1 We can use simple techniques to find the following limits:

(a) $\lim\limits_{x\to 1} \frac{x-1}{x+1} = 0$.

(b) $\lim\limits_{x\to 1} \frac{x^2-1}{x-1} = 2$.

(c) $\lim\limits_{x\to +\infty} \frac{1}{x} = 0$.

(d) $\lim\limits_{x\to 0^+} \frac{1}{x} = +\infty$.

(e) $\lim\limits_{x\to 1} \frac{x^2+x-2}{x^2-x} = 3$.

(f) $\lim\limits_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} = -\frac{1}{3}$.

Theorem 0.2.1 ***(The Sandwich Theorem)*** *Suppose that* $g(x) \le f(x) \le h(x)$ *for all* x *in some open interval containing* c*, except possibly at* $x = c$ *and that*

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L \quad \textit{then} \quad \lim_{x\to c} f(x) = L$$

Example 0.2.2 Suppose that $f(x)$ is a function that satisfies $1 - x^2 \le f(x) \le 1 + x^2$. Then $\lim\limits_{x\to 0} f(x) = 1$ since $\lim\limits_{x\to 0}(1 - x^2) = \lim\limits_{x\to 0}(1 + x^2) = 1$.

Example 0.2.3 Find $\lim\limits_{x\to +\infty} \frac{\sin x}{x}$. Since

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

[1] This is a review of chapter two in the textbook

and $\lim_{x\to\infty} \frac{1}{x} = 0$, then, by the sandwich theorem

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0$$

Remark 0.2.1 Please do not confound the previous limit with $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Example 0.2.4 Consider the function

$$f(x) = \begin{cases} x+1 & , \quad x \leq 0 \\ -x & , \quad x > 0 \end{cases}$$

Then, $\lim_{x\to 0^+} f(x) = 0$ and $\lim_{x\to 0^-} f(x) = 1$. So, $\lim_{x\to 0} f(x)$ does not exist.

0.3 Continuity

Definition 0.3.1 *A function f is continuous at a point x_0 if the following conditions are satisfied:*

(a) $f(x_0)$ *exists.*

(b) $\lim_{x\to x_0} f(x)$ *exists.*

(c) $\lim_{x\to x_0} f(x) = f(x_0)$.

Example 0.3.1 The functions $\sin x, \cos x$, $|x|$ and all polynomials are continuous on $(-\infty, \infty)$.

Example 0.3.2 The rational functions are continuous at all points except at the zeros of the denominator. For example, the function

$$f(x) = \frac{x^3 + x + 1}{x^2 - 1}$$

is continuous on $(-\infty, \infty) \setminus \{-1, 1\}$.

Example 0.3.3 (a function with removable discontinuity) Consider the function

$$f(x) = \frac{x^2 + 2x - 3}{x^2 - 1}$$

Then

$$\lim_{x\to 1} \frac{x^2 + 2x - 3}{x^2 - 1} = \lim_{x\to 1} \frac{(x-1)(x+3)}{(x-1)(x+1)} = \lim_{x\to 1} \frac{x+3}{x+1} = 2$$

The point $x = 1$ is called a **removable discontinuity** of the function f because we can define f at $x = 1$ so that we can remove the discontinuity. The following function is called the **continuous extension of f at** $x = 1$

$$F(x) = \begin{cases} f(x) & , \quad x \neq 1 \\ 2 & , \quad x = 1 \end{cases}$$

Theorem 0.3.1 ***(The intermediate value theorem)*** *If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.*

Recall that a point c is called a root of a function f is $f(c) = 0$. We can use the intermediate value theorem to show that a given function has a root in some interval.

Example 0.3.4 Let $f(x) = x^3 - x - 1$. Since $f(1) = -1 < 0$, $f(2) = 5 > 0$ and $f(1) < 0 < f(5)$ then there exists $c \in [1, 2]$ such that $f(c) = 0$.

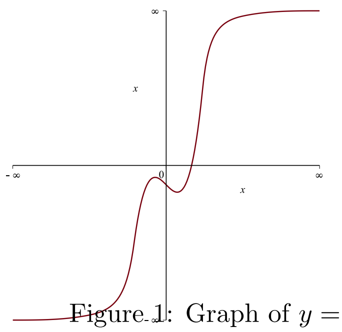

Figure 1: Graph of $y = \frac{1}{x}$

0.3.1 Asymptotes

In this section, we are dealing mainly with rational functions. A rational function is the ratio of two polynomials. Our objective is to be able to sketch some rational functions using limits and asymptotes.

Definition 0.3.2 *A line $y = b$ is a horizontal asymptote of the graph of the function $y = f(x)$ if either*

$$\lim_{x\to\infty} f(x) = b \quad or \quad \lim_{x\to-\infty} f(x) = b$$

Example 0.3.5 The line $y = 0$ is a horizontal asymptote for $f(x) = \frac{x}{x^2+1}$ since $\lim\limits_{x\to+\infty} \frac{x}{x^2+1} = \lim\limits_{x\to-\infty} \frac{x}{x^2+1} = 0$.

Example 0.3.6 The line $y = 1$ is a horizontal asymptote for $f(x) = \frac{x^2}{x^2+1}$ since $\lim\limits_{x\to+\infty} \frac{x^2}{x^2+1} = \lim\limits_{x\to-\infty} \frac{x^2}{x^2+1} = 1$.

Definition 0.3.3 *A line $x = a$ is a vertical asymptote of the graph of the function $y = f(x)$ if either*

$$\lim_{x\to a^+} f(x) = \pm\infty \quad or \quad \lim_{x\to a^-} f(x) = \pm\infty$$

Example 0.3.7 The line $x = 0$ is a vertical asymptote for $f(x) = \frac{1}{x}$ since $\lim_{x\to 0^+} \frac{1}{x} = +\infty$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$.

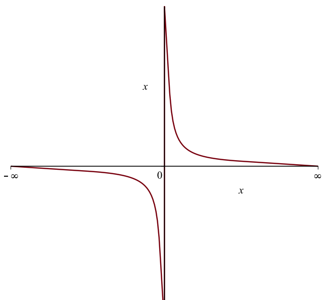

Figure 2: Graph of $y = \frac{1}{x}$

Example 0.3.8 The function $f(x) = \frac{\sin x}{x}$ has no vertical asymptote even it is undefined at $x = 0$ since $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Example 0.3.9 Consider the function $f(x) = \frac{x+1}{x-1}$. Notice that

$$\lim_{x\to 1^+} \frac{x+1}{x-1} = +\infty, \quad \lim_{x\to 1^-} \frac{x+1}{x-1} = +\infty$$

and

$$\lim_{x\to +\infty} \frac{x+1}{x-1} = \lim_{x\to -\infty} \frac{x+1}{x-1} = 1$$

Then the line $x = 1$ is a vertical asymptote and the line $y = 1$ is a horizontal asymptote.

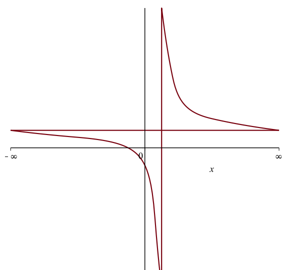

Figure 3: Graph of $y = \frac{x+1}{x-1}$

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator then the graph of f has an **oblique asymptote**.

Example 0.3.10 The graph of the function $f(x) = \frac{x^2}{x-1}$ has an oblique asymptote since the degree of the numerator is 2 and the degree of the denominator is one. Using polynomial division, we can write

$$f(x) = (x+1) + \frac{1}{x-1}$$

So, the line $y = x + 1$ is the oblique asymptote of the graph of f. Moreover, the line $x = 1$ is a vertical asymptote for the graph of f since $\lim_{x \to 1^+} f(x) = +\infty$ and $\lim_{x \to 1^-} f(x) = -\infty$.

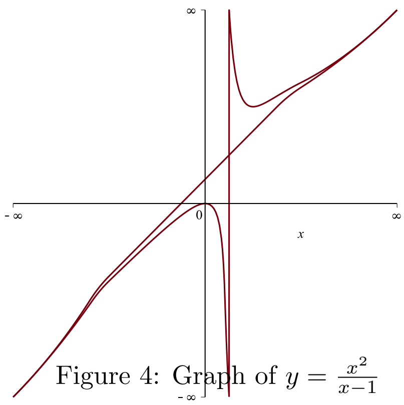

Figure 4: Graph of $y = \frac{x^2}{x-1}$

0.4 Exercises

1. Find the following limits:

   (a) $\lim\limits_{t\to -1} \frac{t^2+3t+2}{t^2-t-2}$

   (b) $\lim\limits_{x\to 1} \frac{1-\sqrt{x}}{1-x}$

   (c) $\lim\limits_{\theta\to 1} \frac{\theta^4-1}{\theta^3-1}$

   (d) $\lim\limits_{\theta\to 0} \frac{\sin(2\theta)}{3\theta}$

   (e) $\lim\limits_{\theta\to 0} \frac{1-\cos\theta}{\sin(2\theta)}$

   (f) $\lim\limits_{x\to \infty} \frac{1+\sqrt{x}}{1-\sqrt{x}}$

   (g) $\lim\limits_{x\to -\infty} \frac{\sqrt{x^2+1}}{x+1}$

   (h) $\lim\limits_{x\to -\infty} \frac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}}$

   (i) $\lim\limits_{x\to \infty} (\sqrt{x^2+1}-\sqrt{x^2-x})$

   (j) $\lim\limits_{t\to 3^+} \frac{\lfloor t\rfloor}{t}$

   (k) $\lim\limits_{x\to 0} x\sin\left(\frac{1}{x}\right)$

2. Find the asymptotes of the following functions then sketch their graphs

   (a) $f(x) = \frac{x+1}{x-1}$

   (b) $y = \frac{x^3+1}{x^2}$

   (c) $f(x) = \frac{x^2+1}{x-1}$

   (d) $f(x) = \frac{x^3+1}{x^2-1}$

3. For what values of a and b is

$$g(x) = \begin{cases} ax+2b & , \quad x \le 0 \\ x^2+3a-b & , \quad 0 < x \le 2 \\ 3x-5 & , \quad x > 2 \end{cases}$$

   continuous at every x. Then sketch the graph of the function.

4. Find the continuous extension of the function $h(t) = \frac{t^2+3t-10}{t-2}$.

5. Use the intermediate value theorem to show that the function $f(x) = x^3-2x^2+2$ has a root.