Lectures 4 and 5: Review of Differentiation

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September 17, 2015

0.1 Differentiation

[1]

0.2 Definition of derivative

Definition 0.2.1 *The derivative of a function f at x_0, denoted $f'(x_0)$ is*

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists.

If $f'(x_0)$ exists then we say that f is **differentiable** at x_0. When we say that f is differentiable on a closed interval $[a, b]$, we mean the following

- f' exists at all points in the open interval (a, b).
- The **right-hand derivative of f at a** exists; that is,

  $$\lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h}$$

  exists. We denote the right-hand derivative of f at $x = a$ by $f'_+(a)$.
- The **left-hand derivative of f at b** exists; that is,

  $$\lim_{h \to 0^-} \frac{f(a + h) - f(a)}{h}$$

  exists. We denote the left-hand derivative of f at $x = b$ by $f'_-(b)$.

Remark 0.2.1 A function f is differentiable at $x = c$ if and only if the right-hand derivative and the left-hand derivative both exist and are equal at $x = c$.

If f is differentiable at $x = c$ then f is continuous at $x = c$. The converse of this statement is not true, the function $f(x) = |x|$ is continuous but not differentiable at $x = 0$.

Example 0.2.1 Let $f(x) = |x|$. We find the left-hand and right-hand derivatives of f at $x = 0$.

$$f'_+(0) = \lim_{h \to 0^+} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$

$$f'_-(0) = \lim_{h \to 0^-} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$$

We conclude that f is not differentiable at $x = 0$.

[1] This is a review of chapter 3 in the textbook

0.3 Differentiation rules

Theorem 0.3.1 *Suppose that $f(x)$ and $g(x)$ are differentiable at x. Then*

1. $(f(x) \pm g(x))' = f'(x) \pm g'(x)$.
2. $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.
3. $\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}$.
4. $(f \circ g)'(x) = f'(g(x))g'(x)$ *(Chain Rule).*

0.4 Derivatives of Trigonometric functions

1. $(\sin x)' = \cos x$.
2. $(\cos x)' = -\sin x$.
3. $(\tan x)' = \sec^2 x$.
4. $(\sec x)' = \sec x \tan x$.
5. $(\csc x)' = -\csc x \cot x$.
6. $(\cot x)' = -\csc^2 x$.

Example 0.4.1 Find the derivatives of the following functions:

1. $\frac{d}{dx}\frac{x+1}{x^2+1} = \frac{x^2+1-(x+1)(2x)}{(x^2+1)^2} = \frac{1-2x-x^2}{(x^2+1)^2}$.
2. $\frac{d}{dx}\tan(\sqrt{x}) = (\sec^2\sqrt{x})\frac{1}{2\sqrt{x}}$.
3. $\frac{d}{dx}(\sec x \tan x) = \sec^3 x + \sec x \tan^2 x$.

Example 0.4.2 Find the equation of the tangent line to the curve $f(x) = \sec x \tan x$ at $x = \frac{\pi}{4}$.

Solution: The slope of the tangent line is $f'(\frac{\pi}{4}) = 3\sqrt{2}$ (from the above example) and $f(\frac{\pi}{4}) = \sqrt{2}$.

Then, the equation of the tangent line to $f(x)$ at $x = \frac{\pi}{4}$ is

$$y - \sqrt{2} = 3\sqrt{2}(x - \frac{\pi}{4})$$

0.4.1 Implicit differentiation

In this section, we consider equations that define relation between x and y. We will learn how to find $\frac{dy}{dx}$ using implicit differentiation. Let us consider some examples:

Example 0.4.3 The equation $x^2 + y^2 = 1$ defines the unit circle (the circle with center $(0, 0)$ and radius one). To find y', we differentiate both sides with respect to x to get $2x + 2yy' = 0$, from which we find that $y' = -x/y$.

Example 0.4.4 Consider the implicit equation $xy = \cot(xy)$. Differentiate both sides with respect to x. Then

$$y + xy' = -\csc^2(xy)(y + xy')$$

From which we find that

$$\frac{dy}{dx} = \frac{-y - y\csc^2(xy)}{x + x\csc^2(xy)}$$

0.5 Linearization and Differentials

Sometimes, we need to approximate a given nonlinear function with a linear function at some point near $(a, f(a))$. The best linear function that approximates $f(x)$ near $x = a$, provided that f is differentiable at $x = a$, is its tangent line whose equation is given by

$$L(x) = f(a) + f'(a)(x - a)$$

$L(x)$ is called the **linearization of** $f(x)$ **at** $x = a$ and the approximation $f(x) \approx L(x)$ is called the **standard linear approximation of** f **at** a.

Example 0.5.1 The linearization of the function $f(x) = \sqrt{1 + x}$ at $x = 0$ is $L(x) = 1 + \frac{1}{2}x$. We can use the linearization to approximate the values of f near $x = 0$. For example, $\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.1$ and $\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$.

Example 0.5.2 Find the linearization of the function $f(x) = \sec x$ at $x = \frac{\pi}{4}$. We need to find $f(\frac{\pi}{4})$ and $f'(\frac{\pi}{4})$. Now, $f'(x) = \sec x \tan x$, so $f'(\frac{\pi}{4}) = \sqrt{2}$ and $f(\frac{\pi}{4}) = \sqrt{2}$. Then the linearization $L(x) = \sqrt{2} + \sqrt{2}(x - \frac{\pi}{4})$.

Now, suppose that we move from a point $x = a$ to a nearby point $a + dx$. The change in f is $\Delta f = f(a + dx) - f(a)$ while the change in L is

$$\Delta L = L(a + dx) - L(a) = f(a) + f'(a)(a + dx - a) - f(a) = f'(a)dx$$

Since $f \approx L$ then $\Delta f \approx \Delta L = f'(a)dx$. Therefore, $f'(a)dx$ gives an approximation for Δf. The quantity $f'(a)dx$ is called the **differential of** f **at** $x = a$. For example, the differential of the function $y = \tan^2 x$ is $dy = 2\tan x \sec^2 x dx$.

Example 0.5.3 The radius r of a circle increases from 10 to 10.1 m. Use dA to estimate the increase in the circle's area A. Estimate the area of the enlarged circle and compare your estimate to the true area found by direct calculations.
Solution The area of the circle is $A = \pi r^2$. Then $dA = 2\pi r dr$. The estimated increase is

$$dA = 2\pi(10)0.1 = 2\pi m^2$$

The estimate area of the enlarged circle is

$$A(10.1) \approx A(10) + dA = 100\pi + 2\pi = 102\pi$$

The exact value of the area is $A(10.1) = \pi(10.1)^2 = 102.01\pi$. The error in this estimation is $|102.01\pi - 102\pi| = 0.01\pi$.

0.6 Exercises

1. Find the derivatives of the following functions:
   (a) $f(s) = \frac{\sqrt{s}-1}{\sqrt{s}+1}$
   (b) $f(x) = (\frac{1}{x} - x)(x^2 + 1)$
   (c) $g(x) = \sec(2x + 1)\cot(x^2)$
   (d) $s(t) = \frac{1+\csc t}{1-\csc t}$
   (e) $f(x) = x^3 \sin x \cos x$.
   (f) $x^{1/2} + y^{1/2} = 1$.
2. Find $\frac{dy}{dx}$ for the following:
   (i) $y = \cot\theta$
   (ii) $x^2 + y^2 = x$.
   (iii) $y = \frac{\sin t}{1-\cos t}$.
3. Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is parallel to the $x-$axis.
4. For what values of the constant a, if any, is

$$f(x) = \begin{cases} \sin(2x) & , \quad x \leq 0 \\ ax & , \quad x > 0 \end{cases}$$

   (i) continuous at $x = 0$?
   (ii) Differentiable at $x = 0$.
5. Find the normals to the curve $xy + 2x - y = 0$ that are parallel to the line $2x + y = 0$.

6. Find the linearization of the following functions at the given points

   (a) $f(x) = \tan x$, $x = \pi/4$.

   (b) $g(x) = \frac{1}{x}$, $x = 1$.

   (c) $h(x) = \frac{x^2}{x^2+1}$, $x = 0$.

   (d) $f(x) = 1 + \cos\theta$, $\theta = \frac{\pi}{3}$.

7. The radius of a circle is increased from 2 to 2.02 m.

   (a) Estimate the resulting change in area.

   (b) Express the estimate as a percentage of the circle's original area.