

## Ch.2 :- Limits and Continuity

Q<sub>1</sub> Find the following limits :-

$$\textcircled{a} \quad \lim_{t \rightarrow -1} \frac{t^2 + 3t + 2}{t^2 - t - 2} = \frac{0}{0}$$

$$\lim_{t \rightarrow -1} \frac{(t+1)(t+2)}{(t+1)(t-2)} = \lim_{t \rightarrow -1} \frac{t+2}{t-2} = \frac{1}{-3}$$

$$\textcircled{b} \quad \lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x} = \frac{0}{0}$$

$$\lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{(1 - \sqrt{x})(1 + \sqrt{x})} = \lim_{x \rightarrow 1} \frac{1}{1 + \sqrt{x}} = \frac{1}{2}$$

$$\textcircled{c} \quad \lim_{x \rightarrow 1} \frac{x^4 - 1}{x^3 - 1} = \frac{0}{0}$$

$$\lim_{x \rightarrow 1} \frac{(x^2 - 1)(x^2 + 1)}{(x - 1)(x^2 + x + 1)} = \lim_{x \rightarrow 1} \frac{(x+1)(x-1)(x^2 + 1)}{(x-1)(x^2 + x + 1)} = \frac{4}{3}$$

$$\textcircled{d} \quad \lim_{\theta \rightarrow 0} \frac{\sin 2\theta}{3\theta} = \frac{1}{3} \cdot 2 \lim_{\theta \rightarrow 0} \frac{\sin 2\theta}{2\theta} = \frac{2}{3} \cdot 1 = \frac{2}{3}$$

OR

$$\lim_{\theta \rightarrow 0} \frac{\frac{\sin 2\theta}{\theta}}{\frac{3\theta}{\theta}} = \frac{\lim_{\theta \rightarrow 0} \frac{\sin 2\theta}{\theta}}{\lim_{\theta \rightarrow 0} 3} = \frac{2}{3}$$

$$(e) \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin(2\theta)} = \frac{0}{0}$$

$$\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin(2\theta)} \cdot \frac{1 + \cos \theta}{1 + \cos \theta} = \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{2 \sin \theta \cos \theta (1 + \cos \theta)}$$

$$= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{2 \cos \theta (1 + \cos \theta)}$$

$$= \frac{0}{2(1+1)} = \frac{0}{4} = 0$$

OR

$$\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin(2\theta)} = \frac{\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta}}{\lim_{\theta \rightarrow 0} \frac{\sin(2\theta)}{\theta}} = \frac{0}{2} = 0$$

$$(f) \lim_{x \rightarrow 0} \frac{1 + \sqrt{x}}{1 - \sqrt{x}} = \frac{\sqrt{1}}{-\sqrt{1}} = \boxed{-1}$$

$$(g) \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 1}}{x + 1}$$

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 (1 + \frac{1}{x^2})}}{x (1 + \frac{1}{x})}$$

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2} \sqrt{1 + \frac{1}{x^2}}}{x (1 + \frac{1}{x})}$$

Note:-

$$\sqrt{x^2} = |x|$$

$$(\sqrt{x})^2 = x$$

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$$\lim_{x \rightarrow -\infty} \frac{|x| \sqrt{1 + \frac{1}{x^2}}}{x \left(1 + \frac{1}{x}\right)}$$

$$\lim_{x \rightarrow -\infty} \frac{-x \sqrt{1 + \frac{1}{x^2}}}{x \left(1 + \frac{1}{x}\right)} = \boxed{-1}$$

(2)  $\lim_{x \rightarrow +\infty} \left( \sqrt{x^2+1} - \sqrt{x^2-x} \right)$  Multiplying by the conjugate

$$\lim_{x \rightarrow \infty} \left( \sqrt{x^2+1} - \sqrt{x^2-x} \right) \left( \frac{\sqrt{x^2+1} + \sqrt{x^2-x}}{\sqrt{x^2+1} + \sqrt{x^2-x}} \right)$$

$$\lim_{x \rightarrow \infty} \frac{x+1}{\sqrt{x^2+1} + \sqrt{x^2-x}} = \lim_{x \rightarrow \infty} \frac{x \left(1 + \frac{1}{x}\right)}{|x| \sqrt{1 + \frac{1}{x^2}} + |x| \sqrt{1 - \frac{1}{x}}}$$

$$= \lim_{x \rightarrow \infty} \frac{x \left(1 + \frac{1}{x}\right)}{|x| \left( \sqrt{1 + \frac{1}{x^2}} + \sqrt{1 - \frac{1}{x}} \right)}$$

$$= \lim_{x \rightarrow \infty} \frac{x \left(1 + \frac{1}{x}\right)}{x \left( \sqrt{1 + \frac{1}{x^2}} + \sqrt{1 - \frac{1}{x}} \right)}$$

$$= \frac{1}{1 + 1} = \frac{1}{2}$$

$$(h) \lim_{x \rightarrow \infty} \frac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}} = \frac{\sqrt[3]{1} - \sqrt[5]{1}}{\sqrt[3]{1} + \sqrt[5]{1}} = \frac{1 - 1}{1 + 1} = 0$$

$$(j) \lim_{t \rightarrow 3^+} \frac{\lfloor t \rfloor}{t}$$

[H] : Greatest Integer function

$$= \frac{\lfloor 3.1 \rfloor}{3} = \frac{3}{3} = 1$$

$$(k) \lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right)$$

By Sandwich Th

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$-|x| \leq |x| \sin\left(\frac{1}{x}\right) \leq |x|$$

$$0 = \lim_{x \rightarrow 0} -|x| \leq \lim_{x \rightarrow 0} |x| \sin\left(\frac{1}{x}\right) \leq \lim_{x \rightarrow 0} |x| = 0$$

Then By Sandwich Th.  $\lim_{x \rightarrow 0} |x| \sin\left(\frac{1}{x}\right) = 0$

$$\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = 0$$

OR  $\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right)$

$= \lim_{x \rightarrow 0} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$

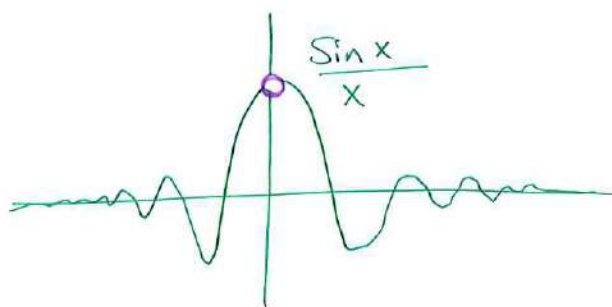
Assume  $y = \frac{1}{x}$

$x \rightarrow 0$

$y \rightarrow \infty$  or  $-\infty$

$= \lim_{y \rightarrow \pm\infty} \frac{\sin y}{y}$

$= 0$



**Q2** Find the asymptotes of the following functions

Summary :-

(a)  $f(x) = \frac{x+1}{x-1}$

Asymptotes :-

Horizontal Asy

$y = b$

$\deg g(x) < \deg h(x)$

$\lim_{x \rightarrow \infty} f(x) = b$

OR  $\lim_{x \rightarrow -\infty} f(x) = b$

Vertical Asy.

$x = a$

Check zero of Denominator

$\lim_{x \rightarrow a^+} f(x) = \pm\infty$

OR  $\lim_{x \rightarrow a^-} f(x) = \pm\infty$

Oblique Asy

$y = ax + b$

$\deg g(x) > \deg h(x)$

Long Division

a)  $f(x) = \frac{x+1}{x-1}$  ,  $\boxed{x \neq 1}$

O. Asy: None

H. Asy: Check  $\lim_{x \rightarrow \pm\infty} f(x)$

$$\lim_{x \rightarrow \infty} \frac{x+1}{x-1} = 1$$

H. Asy.  
 $\boxed{y=1}$

OR  $\lim_{x \rightarrow -\infty} \frac{x+1}{x-1} = 1$

V. Asy: Check  $\lim_{x \rightarrow 1} f(x)$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{x+1}{x-1} = \frac{2}{\text{Small}^+} = \infty$$

V. Asy.  
 $\boxed{x=1}$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \frac{x+1}{x-1} = \frac{2}{\text{Small}^-} = -\infty$$

for the graph.

- X intercept

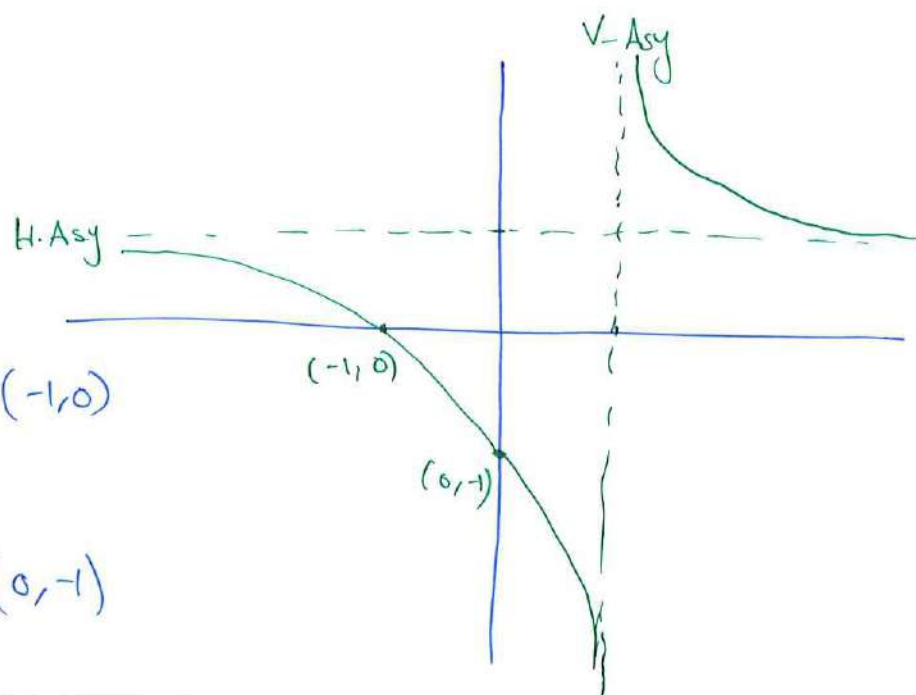
$$0 = \frac{x+1}{x-1}$$

$$0 = x+1 \rightarrow x = -1 \quad (-1, 0)$$

- y-intercept

$$y = \frac{0+1}{0-1}$$

$$y = -1 \quad (0, -1)$$



$$b) \quad y = \frac{x^3 + 1}{x^2}, \quad \boxed{D. \quad x \neq 0}$$

H. Asy : None

O. Asy :  $\boxed{y = x}$

$$\frac{x^2 \sqrt{x^3 + 1}}{-x^3}$$

$$\boxed{\frac{x^3 + 1}{x^2} = x + \frac{1}{x^2}}$$

V. Asy : check  $\lim_{x \rightarrow 0} \frac{x^3 + 1}{x^2}$

$$\lim_{x \rightarrow 0^+} \frac{x^3 + 1}{x^2} = \frac{1}{\text{Small}^+} = \infty$$

$$\lim_{x \rightarrow 0^-} \frac{x^3 + 1}{x^2} = \frac{1}{\text{Small}^-} = -\infty$$

V. Asy.  
 $\boxed{x = 0}$

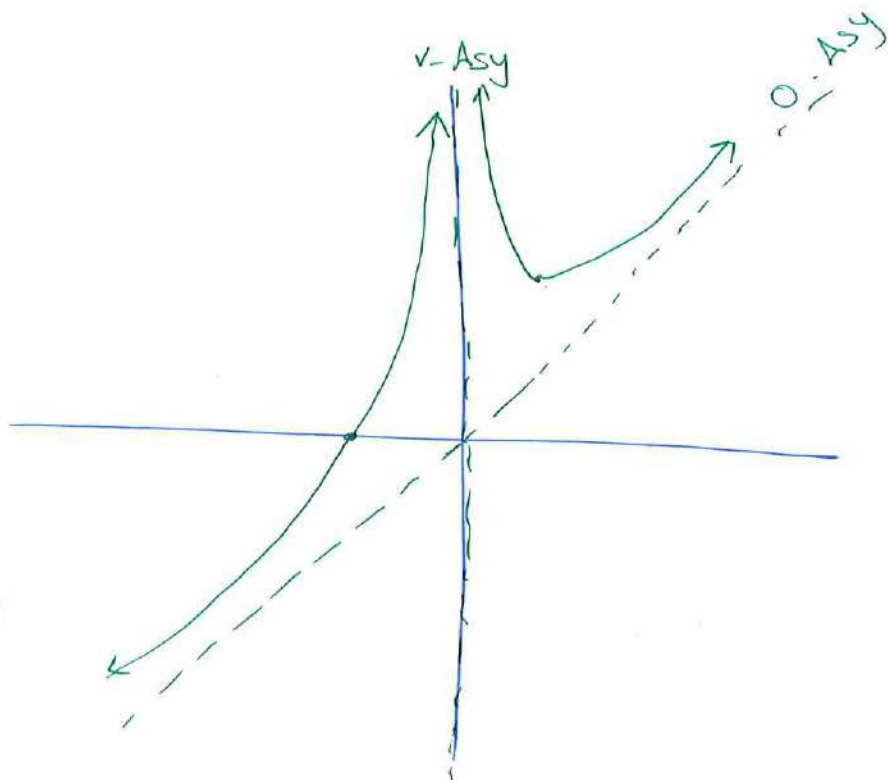
for the graph.

- x-intercept  $0 = \frac{x^3 + 1}{x^2}$

$$\begin{aligned} x^3 + 1 &= 0 \\ x^3 &= -1 \rightarrow x = -1 \\ &(-1, 0) \end{aligned}$$

y-intercept  $y = \frac{1}{0}$  DNE

No y-intercept.



$$\textcircled{c} f(x) = \frac{x^2+1}{x-1} \quad \text{D. } \boxed{x \neq 1}$$

H. Asy : None

O. Asy  $\rightarrow \boxed{y = x+1}$

$$\begin{array}{r} x+1 \\ \hline x^2+1 \\ -x^2+x \\ \hline x+1 \\ -x+1 \\ \hline 2 \end{array}$$

$$\frac{x^2+1}{x-1} = \overset{\text{linear}}{\boxed{x+1}} + \frac{2}{x-1}$$

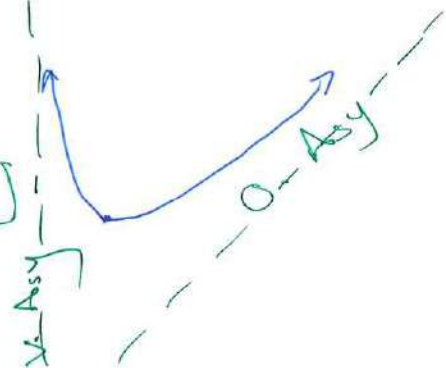
O. Asy

V. Asy : Check  $\lim_{x \rightarrow 1} f(x)$

$$\lim_{x \rightarrow 1^+} \frac{x^2+1}{x-1} = \frac{2}{\text{Small}^+} = \infty$$

$$\lim_{x \rightarrow 1^-} \frac{x^2+1}{x-1} = \frac{2}{\text{Small}^-} = -\infty$$

V. Asy  $\boxed{x=1}$



for the graph

x-intercept  $0 = \frac{x^2+1}{x-1}$

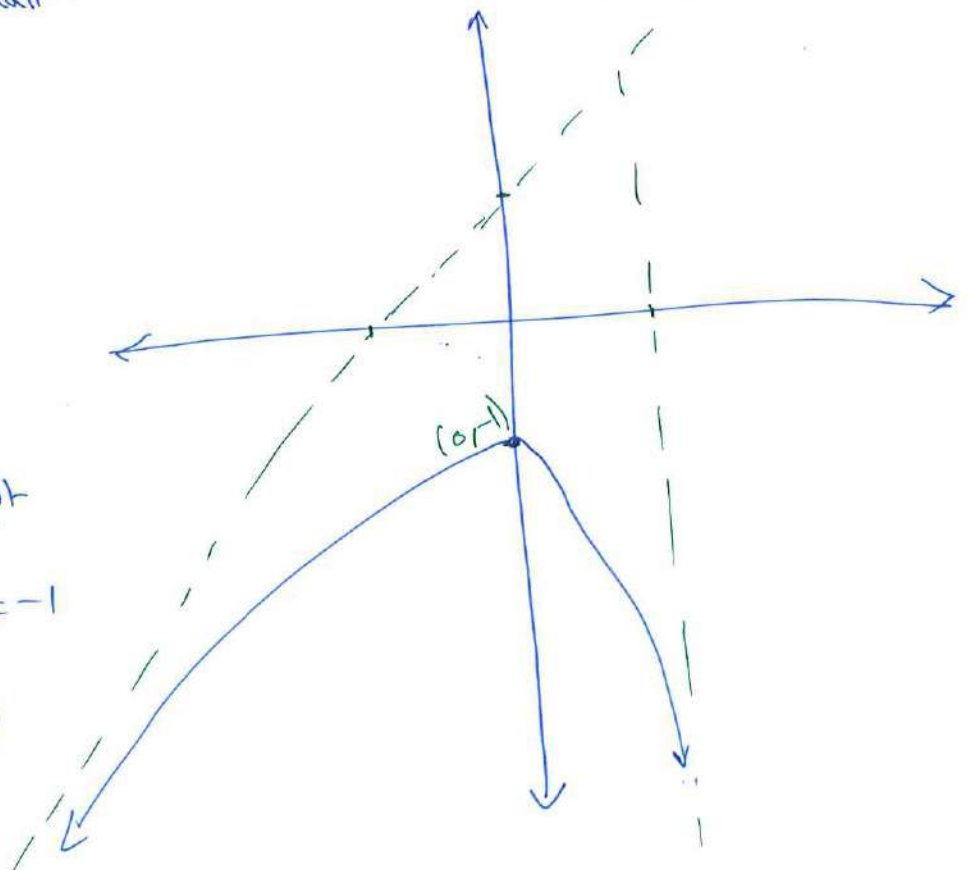
$$x^2+1=0$$

No x-intercept

y-intercept

$$y = \frac{1}{-1} = -1$$

$(0, -1)$





d)  $f(x) = \frac{x^3 + 1}{x^2 - 1}$ ,  $x \neq \pm 1$

$$x^2 - 1 \overline{) \begin{array}{r} x \\ x^3 + 1 \\ -x^3 + x \\ \hline x + 1 \end{array}}$$

• H. Asy : None

• O. Asy :  $y = x$

$$\frac{x^3 + 1}{x^2 - 1} = \underbrace{x}_{\text{O. Asy}} + \frac{x + 1}{x^2 - 1}$$

• V. Asy : Check  $\lim_{x \rightarrow 1} f(x)$  and  $\lim_{x \rightarrow -1} f(x)$

$$\lim_{x \rightarrow 1^+} \frac{x^3 + 1}{x^2 - 1} = \frac{2}{\text{Small } +} = \infty$$

$$\lim_{x \rightarrow 1^-} \frac{x^3 + 1}{x^2 - 1} = \frac{2}{\text{Small } -} = -\infty$$

V. Asy  
 $x = 1$

$$\lim_{x \rightarrow -1} \frac{x^3 + 1}{x^2 - 1} = \frac{0}{0} \rightarrow x = -1 \text{ Removable Discontinuity}$$

$$\lim_{x \rightarrow -1} \frac{\cancel{(x+1)}(x^2 + x + 1)}{\cancel{(x+1)}(x-1)} = \frac{3}{-2}$$

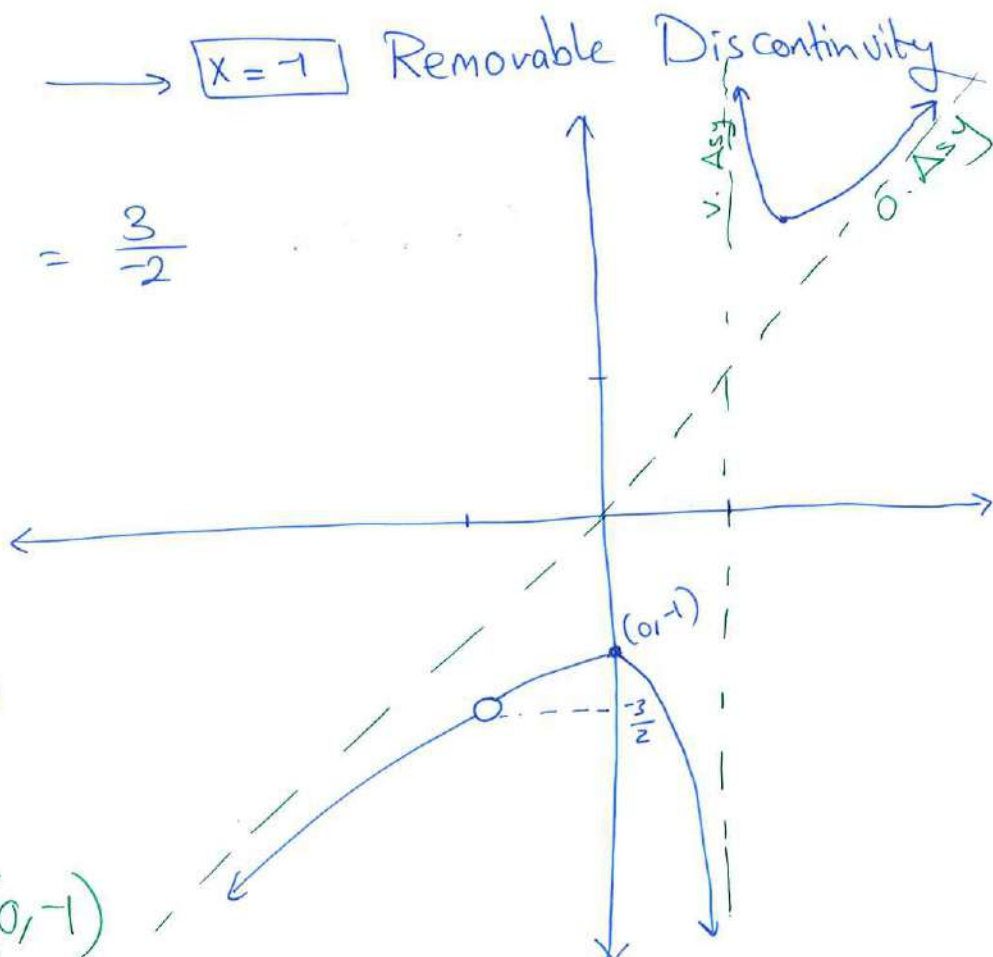
for the graph

x-intercept:  $0 = \frac{x^3 + 1}{x^2 - 1}$

$x^3 + 1 = 0$   
 $x = -1 \notin D$

Reject

y-intercept  $y = \frac{1}{-1}$   
 $y = -1$   $(0, -1)$



**Q<sub>3</sub>** Find the value of a/b ??

$$g(x) = \begin{cases} ax + 2b & x \leq 0 \\ x^2 + 3a - b & 0 < x \leq 2 \\ 3x - 5 & x > 2 \end{cases} \quad g(x) \text{ is cont. everywhere}$$

Then sketch the graph of the function

\* \* Since  $g(x)$  is cont at  $x=0$

$$\lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^-} g(x)$$

$$0 + 3a - b = 2b \rightarrow \boxed{a = b}$$

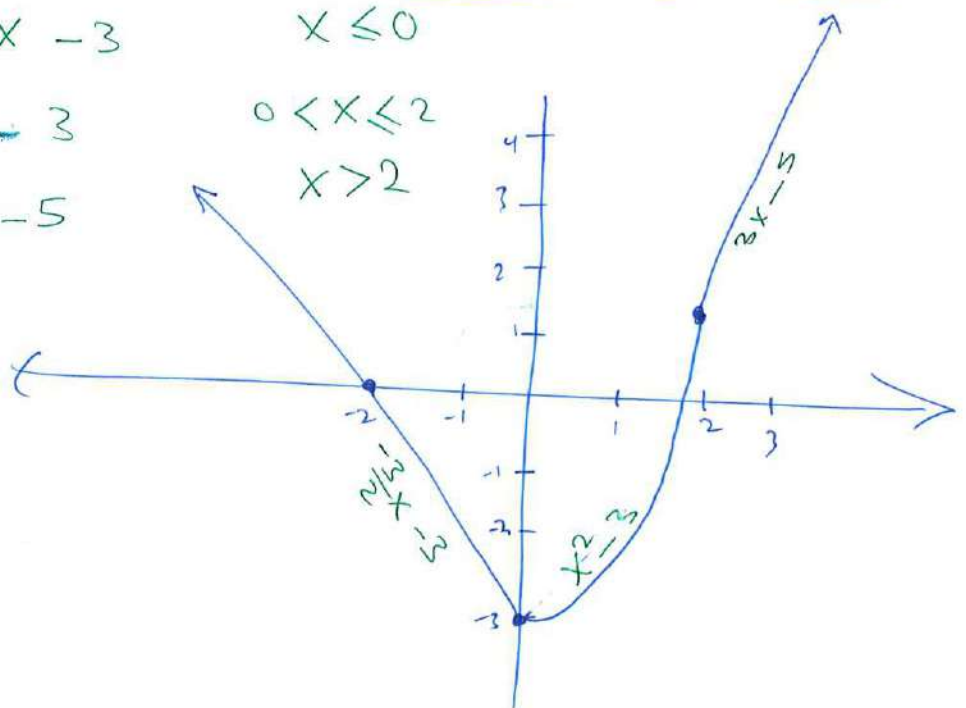
\* \* Since  $g(x)$  is cont at  $x=2$

$$\lim_{x \rightarrow 2^+} g(x) = \lim_{x \rightarrow 2^-} g(x)$$

$$1 = 4 + 3a - b \rightarrow 3a - a = -3$$

$$\boxed{b = \frac{-3}{2}} \text{ , } \boxed{a = \frac{-3}{2}}$$

$$g(x) = \begin{cases} \frac{-3}{2}x - 3 & x \leq 0 \\ x^2 - 3 & 0 < x \leq 2 \\ 3x - 5 & x > 2 \end{cases}$$



## Q4) Continuous Extension

$$h(x) = \frac{x^2 + 3x - 10}{x - 2}, \quad [x \neq 2]$$

$h(2) = \frac{0}{0} \rightarrow [x=2]$  Removable Discontinuity

$$\lim_{x \rightarrow 2} \frac{x^2 + 3x - 10}{x - 2} = \lim_{x \rightarrow 2} \frac{(x-2)(x+5)}{x-2} = 7$$

The the continuous Extension of  $h(x)$  at  $x=2$

$$H(x) = \begin{cases} \frac{x^2 + 3x - 10}{x - 2} & x \neq 2 \\ 7 & x = 2 \end{cases}$$

Q5) Use the Intermediate Value Theorem :- I.V.T to show that the  $f(x) = x^3 + 2x^2 + 2$  has a root (solution)

Conditions :-

①  $f(x)$  is cont. for all  $x$  because its a polynomial  
 $\rightarrow f(x)$  is cont on  $[-2, 2]$

②  $y_0 = 0$  lie between  $f(-2)$  and  $f(2)$

$$f(-2) = -14 \quad f(-2) \leq y_0 \leq f(2)$$

$$f(2) = 2$$

$\rightarrow$  Then by I.V.T,  $\exists c \in [-2, 2]$  such that  $f(c) = y_0 = 0$

$\rightarrow$  This means that  $c$  is a root (solution) for the function