

Ch. 3 Differentiation

Q1. Find the derivative $\frac{dy}{dx}$??

a) $f(x) = \frac{\sqrt{x} - 1}{\sqrt{x} + 1}$

$$f'(x) = \frac{(\sqrt{x} + 1) \cdot \frac{1}{2\sqrt{x}} - (\sqrt{x} - 1) \cdot \frac{1}{2\sqrt{x}}}{(\sqrt{x} + 1)^2}$$

$$= \frac{\frac{\sqrt{x} + 1 - \sqrt{x} + 1}{2\sqrt{x}}}{(\sqrt{x} + 1)^2} = \frac{2}{2\sqrt{x} (\sqrt{x} + 1)^2} = \frac{1}{\sqrt{x} (\sqrt{x} + 1)^2}$$

b) $f(x) = \left(\frac{1}{x} - x\right) (x^2 + 1)$

$$f'(x) = \left(\frac{1}{x} - x\right) 2x + (x^2 + 1) \cdot \left(-\frac{1}{x^2} - 1\right)$$

$$= 2 - 2x^2 + -1 - x^2 - \frac{1}{x^2} - 1$$

$$= \boxed{-3x^2 - \frac{1}{x^2}}$$

c) $g(x) = \sec(2x+1) \cot(x^2)$

$$g'(x) = \sec(2x+1) \cdot -\csc^2(x^2) * 2x$$

$$+ \cot(x^2) \cdot 2\sec(2x+1) \tan(2x+1) \cdot$$

$$g'(x) = 2 \sec(2x+1) \left[\cot(x^2) \tan(2x+1) - x \csc^2(x^2) \right]$$

$$\textcircled{d} \quad g(t) = \frac{1 + \csc(t)}{1 - \csc(t)}$$

$$= \frac{1 + 1/\sin(t)}{1 - 1/\sin(t)} = \frac{\frac{\sin t + 1}{\sin t}}{\frac{\sin t - 1}{\sin t}}$$

$$g(t) = \frac{\sin(t) + 1}{\sin(t) - 1}$$

$$g'(t) = \frac{(\sin(t) - 1)(\cos(t)) - (\sin(t) + 1) \cdot (\cos(t))}{(\sin(t) - 1)^2}$$

$$= \frac{-2 \cos(t)}{(\sin t - 1)^2}$$

$$\textcircled{e} \quad f(x) = x^3 \sin x \cos x$$

$$f(x) = x^3 \frac{\sin(2x)}{2}$$

$$f'(x) = x^2 \cdot \frac{2}{2} \cos(2x) + \frac{\sin(2x)}{2} \cdot 3x^2$$

$$= x^2 \cos(2x) + \frac{3x^2}{2} \sin(2x)$$

(P)

$$x^{1/2} + y^{1/2} = 1$$

$$\sqrt{x} + \sqrt{y} = 1$$

$$\sqrt{y} = 1 - \sqrt{x}$$

$$\frac{y'}{2\sqrt{y}} = \frac{-1}{2\sqrt{x}}$$

$$y' = \frac{-2\sqrt{y}}{2\sqrt{x}} = \frac{-\sqrt{y}}{\sqrt{x}} = \frac{\sqrt{x} - 1}{\sqrt{x}}$$

Q2

(i) $y = \cot^2 x$

$$\frac{dy}{dx} = 2 \cot(x) \cdot -\csc^2(x) = -2 \cot x \csc^2 x$$

(ii) $x^2 + y^2 = x$

Note $y = \sqrt{x - x^2}$

$$2x + 2y \frac{dy}{dx} = 1$$

$$2y \frac{dy}{dx} = 1 - 2x$$

$$\frac{dy}{dx} = \frac{1 - 2x}{2\sqrt{x - x^2}}$$

(iii)

$$y = \frac{\sin x}{1 - \cos x}$$

$$\frac{dy}{dx} = \frac{(1 - \cos x) \cdot \cos x - \sin^2 x}{(1 - \cos x)^2}$$

$$\frac{\cos x - \cos^2 x - \sin^2 x}{(1 - \cos x)^2} = \frac{\cos x - 1}{(1 - \cos x)^2}$$

$$= \frac{1}{\cos x - 1}$$

Q₃ Points = ??

Curve $y = 2x^3 - 3x^2 - 12x + 20$

Where the tangent is parallel to x -axis

Solution

$$y' = \text{Slope of tangent} = \text{slope of } x\text{-axis} = 0$$

$$y' = 6x^2 - 6x - 12 = 0$$

$$x^2 - x - 2 = 0$$

$$(x+1)(x-2) = 0$$

$$x = -1$$

$$x = 2$$

Points :- $(-1, 27)$, $(2, 0)$

Q4

$$f(x) = \begin{cases} \sin(2x) & x \leq 0 \\ ax & x > 0 \end{cases}$$

(i) $f(x)$ is continuous at $x=0$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x)$$

$$\lim_{x \rightarrow 0^+} ax = \lim_{x \rightarrow 0^-} \sin(2x)$$

$$0 = 0$$

--- \rightarrow a is any real number

(ii) $f(x)$ is diff. at $x=0$

$$f'(x) = \begin{cases} 2 \cos(2x) & x < 0 \\ a & x > 0 \end{cases}$$

$$f'(0)^+ = f'(0)^-$$

$$a = 2 \cos(0)$$

$$a = 2$$

Extra question

Determine

the equation of the normal line to

the curve

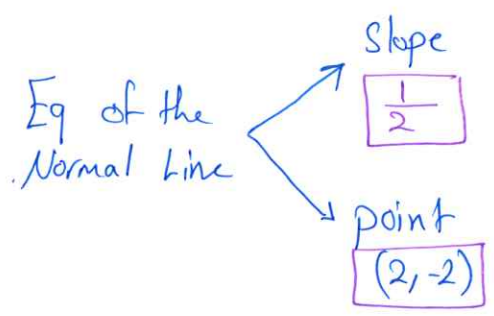
$$4y^2 = x^4$$

at the point $(2, -2)$

$$4y^2 = x^4$$

$$8y y' = 4x^3$$

$$y' = \frac{x^3}{2y} \Big|_{(2,-2)} = \frac{8}{2 \cdot -2} = -2$$



The equation of Normal line

$$y + 2 = \frac{1}{2}(x - 2)$$

$$y = \frac{1}{2}x - 3$$

Q5 Find the normals to the curve $xy + 2x - y = 0$ that are parallel to the line $2x + y = 0$

The equation of Normal

- Slope -2
- point (x_1, y_1)
 (x_2, y_2)

• The normal line // line: $y = -2x$

slope of normal line = -2

slope of the curve = $\frac{1}{2} = y'$

$$xy + 2x - y = 0 \text{ ----- (1)}$$

$$xy' + y + 2 - y' = 0$$

$$y' = \frac{-2 - y}{x - 1} = \frac{1}{2}$$

$$x - 1 = -y - 2y$$

$$x = -3 - 2y \text{ ----- (2)}$$

To find the points $(x_1, y_1), (x_2, y_2)$ of intersection the line with curve :

Substitute equation (2) in equation (1)

$$xy + 2x - y = 0$$

$$(-3 - 2y)y + 2(-3 - 2y) - y = 0$$

$$-3y - 2y^2 - 6 - 4y - y = 0$$

$$-2y^2 - 8y - 6 = 0$$

$$y^2 + 4y + 3 = 0$$

$$(y + 1)(y + 3) = 0$$

$$\boxed{y = -1} \text{ OR } \boxed{y = -3}$$

$$\boxed{x = -1}$$

$$\boxed{x = 3}$$

So the Intersection points between the normal line and the curve are: $(-1, 1)$ and $(3, -3)$

* The first normal is $y + 1 = -2(x + 1)$

$$\boxed{y = -2x - 3}$$

* The second normal is $y + 3 = -2(x - 3)$

$$\boxed{y = -2x + 3}$$

$$\boxed{\text{Q6}} \quad L(x) = f(a) + f'(a)(x - a) \quad \text{linearization of } f(x)$$

$\boxed{x = a}$

$$\textcircled{a} \quad f(x) = \tan x \quad x = \frac{\pi}{4}$$

$$f\left(\frac{\pi}{4}\right) = 1$$

$$f'(x) = \sec^2 x$$

$$f'\left(\frac{\pi}{4}\right) = 2$$

$$\boxed{L(x) = 1 + 2\left(x - \frac{\pi}{4}\right) = 2x - \frac{\pi}{2} + 1}$$

$$\textcircled{b} \quad g(x) = \frac{1}{x} \quad , \quad x=1$$

$$g(1) = 1$$

$$g'(x) = -\frac{1}{x^2}$$

$$g'(1) = -1$$

$$L(x) = 1 + -1(x-1)$$

$$L(x) = 1 - x + 1 = 2 - x$$

$$\textcircled{c} \quad h(x) = \frac{x^2}{x^2+1} \quad , \quad x=0$$

$$h(0) = 0$$

$$h'(x) = \frac{2x}{(x^2+1)^2}$$

$$h'(0) = 0$$

$$L(x) = 0 + 0(x-0) = 0$$

$$\textcircled{d} \quad f(x) = 1 + \cos \theta \quad , \quad \theta = \frac{\pi}{3}$$

$$f\left(\frac{\pi}{3}\right) = \frac{3}{2}$$

$$f'(x) = -\sin \theta$$

$$f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$L(\theta) = \frac{3}{2} + \frac{\sqrt{3}}{2}\left(\theta - \frac{\pi}{3}\right) = \frac{3}{2} - \frac{\sqrt{3}}{2}\theta + \frac{\sqrt{3}\pi}{6}$$

Q7 The radius of a circle increased from 2 → 2.02m

$$A(r) = \pi r^2$$

True change in Area = ΔA

a) Estimate the resulting change in Area.

$$\begin{aligned} \text{Estimated Change in Area} &= dA \\ &= A' dr \\ &= (4\pi) (0.02) \\ &= 0.08 \pi \text{ m}^2 \end{aligned}$$

$$\begin{aligned} dr &= \Delta r \\ &= 0.02 \\ A' &= 2\pi r \\ A' \Big|_{r=2} &= 4\pi \end{aligned}$$

b) Find the True Change in Area.

$$\begin{aligned} \text{True value of Area} &= \Delta A \\ &= A_2 - A_1 \\ &= A(2.02) - A(2) \\ &= 4.0804 \pi - 4 \pi \\ &= 0.0804 \pi \text{ m}^2 \end{aligned}$$

c) Find the $|\Delta A - dA|$

$$\begin{aligned} \text{Error} = |\Delta A - dA| &= |0.0804 \pi - 0.08 \pi| \\ &= 0.0004 \pi \text{ m}^2 \end{aligned}$$

d) Express dA as a percentage of the circle original area

$$\frac{dA}{A} * 100 \% = \frac{0.08 \pi}{4 \pi} * 100 \% = 2 \%$$

