Ch. 4 Application of derivatives

First Derivative Test $f'(x)$

* Critical Value : C.V $\begin{cases} y' = 0 \\ y' \text{ DNE} \end{cases}$

* Increasing/Decreasing : Sign of y'

* Extreme Value : EV $\begin{cases} \text{C.P critical point} \\ \text{E.P endpoint} \end{cases}$

  • And check Them

Second Derivative Test $f''(x)$

* y'' $\begin{cases} y'' = 0 \\ y'' \text{ DNE} \end{cases}$

* Concave up/concave down : Sign of y''

* Inflection point : $y'' = 0$ / y'' DNE

  • And Check them

• If $f(x) = \frac{g(x)}{h(x)}$ check Asymptote lines

Q1 (a) $y = 1 - (x+1)^3$

$y' = -3(x+1)^2$

$y'' = -6(x+1)$

* Domain $D = (-\infty, \infty)$

* C.P $\to$ $y' = 0 \longrightarrow \boxed{x = -1} \in D$

$\to$ y' DNE $\longrightarrow$ None

the critical point is $(-1, 1)$

* Inc / Dec

$- - - - - \underset{-1}{|} - - -$

Interval of decreasing $= (-\infty, \infty)$

Interval of increasing $=$ None

* E.V : f doesn't have Max, Min (No extreme value)

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* Inflection point / Concave up / concave down

y'' $\to$ $y'' = 0 \longrightarrow \boxed{x = -1} \in D$

$\to$ y'' DNE $\longrightarrow$ None

$+ + + + \underset{-1}{|} - - - -$

Concave up $(-\infty, -1]$

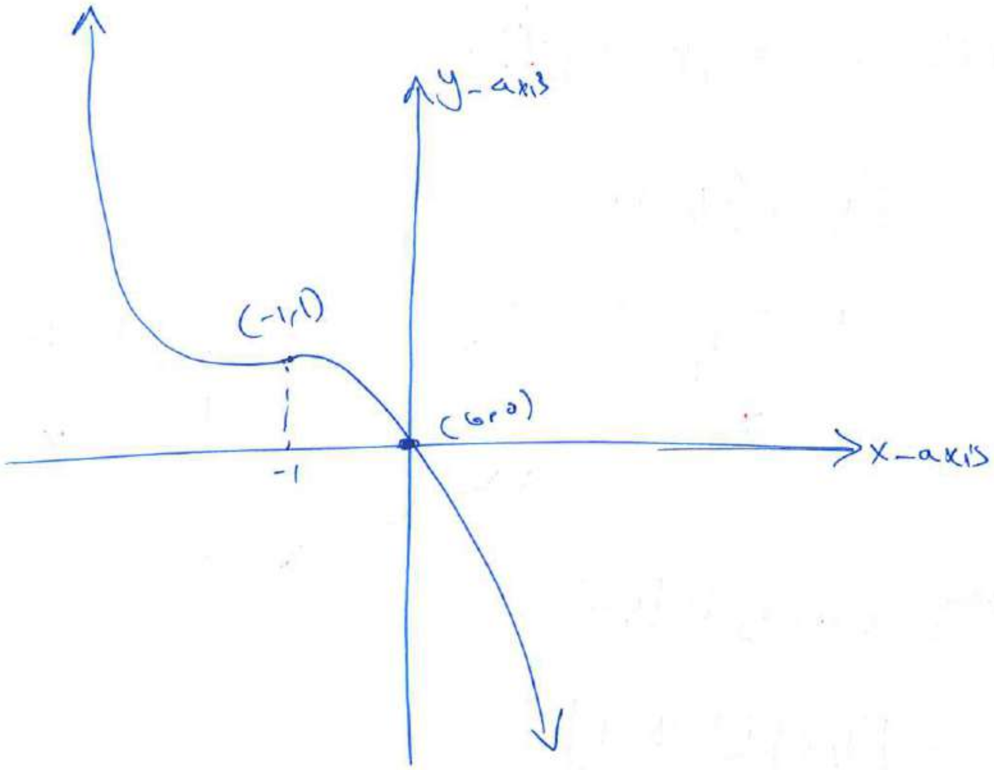
Concave down $[-1, \infty)$

Inflection point $(-1, 1)$

for the graph. $y = 1 - (x+1)^3$

* x-intercept $\longrightarrow$ $0 = 1 - (x+1)^3$ $\longrightarrow$ $\overset{CP}{\boxed{x=0}}$ $(0,0)$

* y-intercept $\longrightarrow$ $y = 1 - 1 = 0$ $\longrightarrow$ $\boxed{y=0}$ $(0,0)$



$D = (-\infty, \infty)$

$(0,0)$	x-intercept y-intercept
$(-1,1)$	Inflection point

(b) $y=\dfrac{x^2+1}{x}$ $\qquad y'=\dfrac{x^2-1}{x^2}$ $\qquad y''=\dfrac{2}{x^3}$

$x \neq 0$

* Domain $\boxed{D=(-\infty,0)\cup(0,\infty)}$

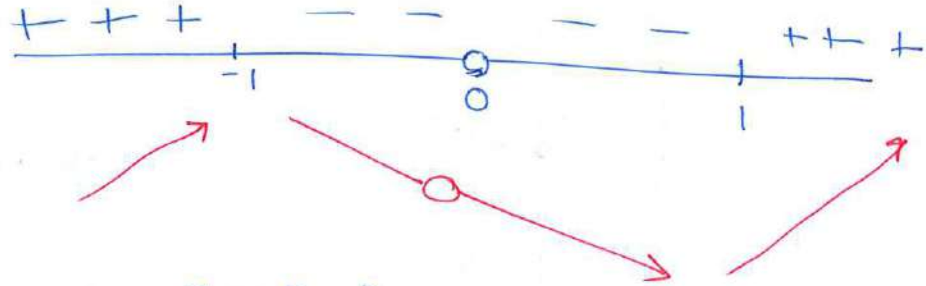
* C.V

$y'=0 \longrightarrow \boxed{x=\pm 1}$ $\in D$

$y'=\text{DNE} \longrightarrow \boxed{x=0}$ $\notin D$

The critical points : $(1,2)$ $(-1,-2)$

* Inc / Dec



Interval of ~~Int~~ Increasing $=(-\infty,-1]\cup[1,\infty)$

Interval of Decreasing $=[-1,0)\cup(0,1]$

* E.V : local Max $=(-1,-2)$

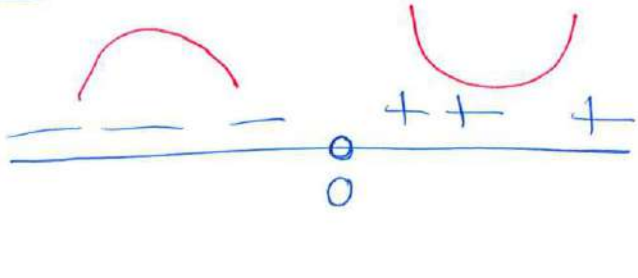
local Min $=(+1,+2)$

* Concave up / concave down.

y'' $\longrightarrow$ $y''=0 \longrightarrow$ None

y'' DNE $\longrightarrow \boxed{x=0}$ $\notin D$

there is no inflection point



Concave down $=(-\infty,0)$

Concave up $=(0,\infty)$

for the graph → x-intercept / y-intercept
→ Asy
→ Max / Min / Inflection point

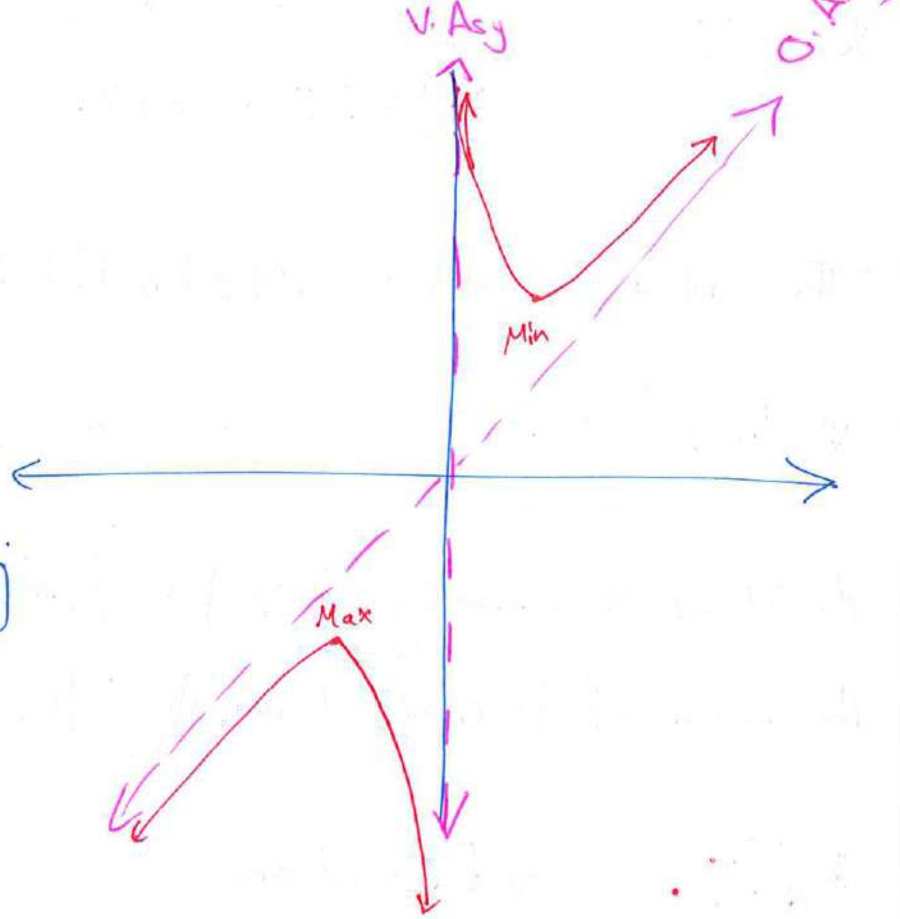
$$y = \frac{x^2+1}{x}, \quad \boxed{x \neq 0}$$

O. Asy: $\boxed{y = x}$

H. Asy: None

V. Asy: check at $x = 0$

$$\left.\begin{aligned} \lim_{x \to 0^+} \frac{x^2+1}{x} &= \infty \\ \lim_{x \to 0^-} \frac{x^2+1}{x} &= -\infty \end{aligned}\right\} \text{V. Asy. } \boxed{x = 0}$$



x-intercept → $y = 0$
$x^2 + 1 = 0$ → No x-intercept

y-intercept → $\boxed{x = 0} \notin D$ → No y-intercept

$x \neq 0$

Max $(-1, -2)$

Min $(1, 2)$

O. Asy $\boxed{y = x}$

V. Asy $\boxed{x = 0}$

$y = x^4 - 2x^2 \qquad y' = 4x(x^2-1) \qquad y'' = 4(3x^2-1)$

* Domain $= (-\infty, \infty)$

* C.V

$y' = 0 \longrightarrow \boxed{x=0}, \boxed{x=1}, \boxed{x=-1}$ ($\in D$, $\in D$, $\in D$)

y' DNE $\longrightarrow$ None

The critical points : $(0,0)$, $(1,-1)$ $(-1,-1)$

* Inc/Dec.

$-$ | -1 | $+$ | 0 | $-$ | 1 | $+$

the interval of Increasing $[-1,0] \cup [1,\infty)$

the interval of Decreasing : $(-\infty,-1] \cup [0,1]$

* E.V $\longrightarrow$ E.P $\to$ None

$\longrightarrow$ C.V

L. Max $= (0,0)$ from the graph   No abs Max

L. Min $= (-1,-1) \longrightarrow$ Abs Min

L. Min $= (1,-1) \longrightarrow$ Abs Min

* Concave up / Concave down

$y'' = 0 \longrightarrow \boxed{x = \pm\frac{1}{\sqrt{3}}}$ $\in D$

y'' DNE

$+ + +$ | $-\frac{1}{\sqrt{3}}$ | $- - -$ | $\frac{1}{\sqrt{3}}$ | $+ +$

Concave up $(-\infty, \frac{-1}{\sqrt{3}}] \cup [\frac{1}{\sqrt{3}}, \infty)$ Concave down $[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]$

Inflection points $\left(\frac{-1}{\sqrt{3}}, \frac{-5}{9}\right)$ & $\left(\frac{1}{\sqrt{3}}, \frac{-5}{9}\right)$

for the graph $\boxed{y = x^4 - 2x^2}$

→ x-intercept $y=0$

$x^4 - 2x^2 = 0$

$x^2(x^2-2) = 0$

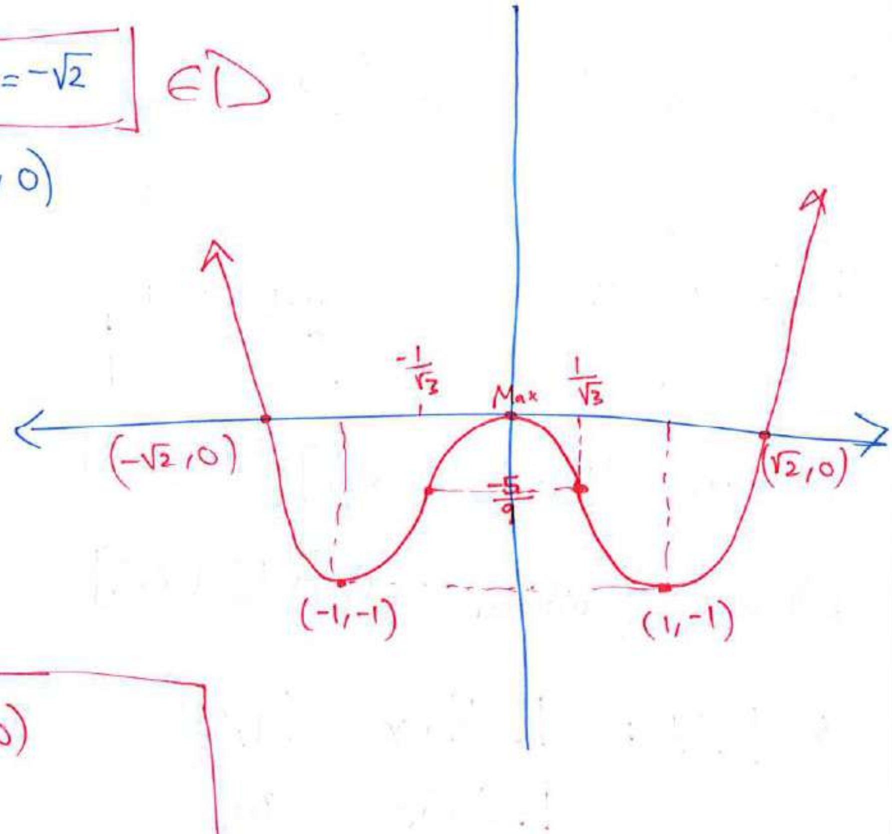
$\boxed{x=0,\ x=\sqrt{2},\ x=-\sqrt{2}}$ ∈D

x-intercept = $(0,0)$, $(\sqrt{2},0)$, $(-\sqrt{2},0)$

→ y-intercept $\boxed{x=0}$ ∈D

$y=0$

y-intercept = $(0,0)$



D: $(-\infty,\infty)$

x-intercept $(0,0)$ $(\sqrt{2},0)$ $(-\sqrt{2},0)$

y-intercept $(0,0)$

Max $(0,0)$

Min $(-1,-1)$ $(1,-1)$

Inflection point $\left(\frac{-1}{\sqrt{3}}, \frac{-5}{9}\right)$ $\left(\frac{1}{\sqrt{3}}, \frac{-5}{9}\right)$

$\boxed{d}$ $y = \dfrac{x^2 - 3}{x - 2}$ $\quad y' = \dfrac{x^2 - 4x + 3}{(x-2)^2}$ $\quad y'' = \dfrac{2}{(x-2)^3}$

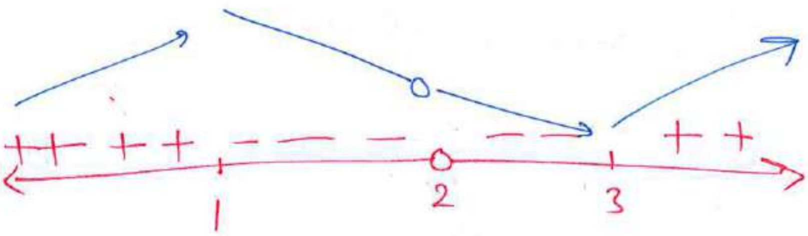
* Domain $= \mathbb{R} \setminus \{2\}$

* C.V $\to$ $y' = 0 \longrightarrow x^2 - 4x + 3 = 0 \to \boxed{x = 1,\ x = 3}$ $\in D$

$\to$ $y' = DNE \longrightarrow (x-2)^2 = 0 \longrightarrow \boxed{x = 2}$ $\notin D$

The critical points $(1,2)$ $(3,6)$

* Inc/Dec



Increasing interval $(-\infty, 1] \cup [3, \infty)$

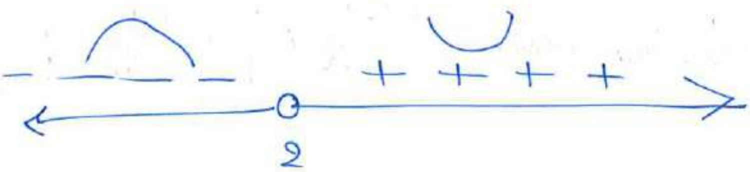
Decreasing interval $[1, 2) \cup (2, 3]$

* E.V : L. Max $(1,2)$

L. Min $(3,6)$

* Concave up/down $\to$ $y'' = 0 \longrightarrow$ None

$\to$ y'' DNE $\longrightarrow \boxed{x = 2}$ $\notin D$



Concave up : $(2, \infty)$

Concave down : $(-\infty, 2)$

Inflection point : None

$$\boxed{y = \frac{x^2 - 3}{x - 2}}$$

ED

x-intercept $\longrightarrow$ $\boxed{y=0}$ $\longrightarrow$ $x^2 - 3 = 0$ $\longrightarrow$ $\boxed{x = \pm\sqrt{3}}$

ED

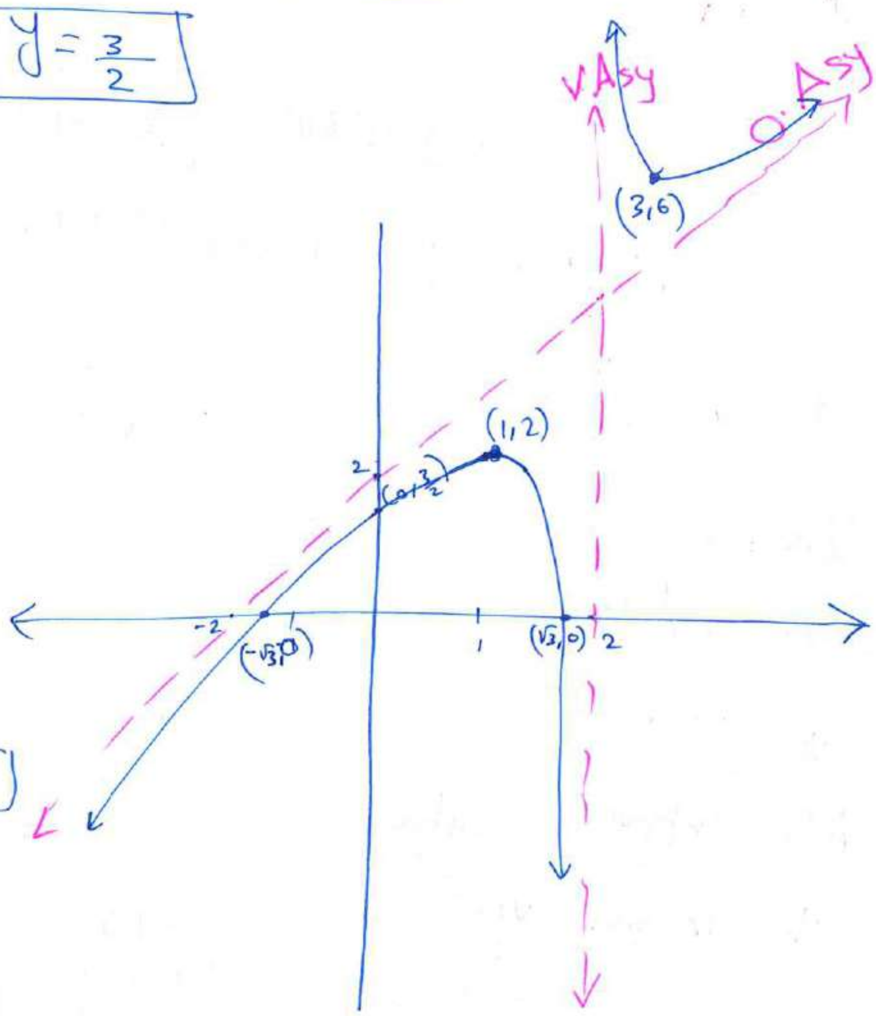
y-intercept $\longrightarrow$ $\boxed{x=0}$ $\longrightarrow$ $\boxed{y = \frac{3}{2}}$

- H. Asy : None
- O. Asy : $\boxed{y = x+2}$
- V. Asy : check at $x = 2$

$\lim_{x \to 2^+} \frac{x^3 + 3}{x - 2} = \infty$

$\lim_{x \to 2^-} \frac{x^3 - 3}{x - 2} = -\infty$

V. Asy $\boxed{x = 2}$



D: $x \neq 2$

x-intercept	$(\sqrt{3}, 0)$ $(-\sqrt{3}, 0)$
y-intercept	$(0, \frac{3}{2})$
O. Asy	$y = x+2$
V. Asy	$x = 2$
Max	$(1, 2)$
Min	$(3, 6)$

(e) $y = \sqrt[3]{x^3+1}$ $\quad y' = \dfrac{x^2}{\sqrt[3]{(x^3+1)^2}}$ $\quad y'' = \dfrac{2x}{(x^3+1)^{\frac{5}{3}}}$

* Domain $(-\infty, \infty)$

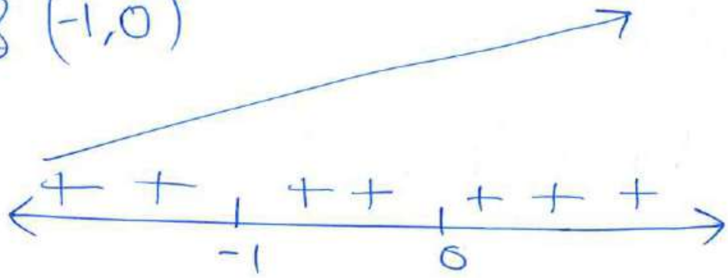
* C.V

$\rightarrow y' = 0 \quad \overset{\in D}{\boxed{x = 0}}$

$\rightarrow y' = \text{DNE} \quad \overset{\in D}{\boxed{x = -1}}$

The critical point $(0,1)$ & $(-1,0)$

* Inc / Dec



Inc: $(-\infty, \infty)$

Dec: None

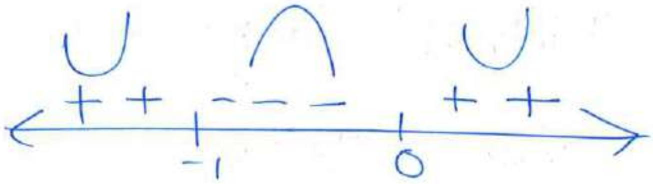
* E.V

No extreme value

* Concave up/down

$y'' \rightarrow y'' = 0 \quad \overset{\in D}{\boxed{x = 0}}$

$y'' \rightarrow y''\ \text{DNE} \quad \overset{\in D}{\boxed{x = -1}}$



Concave up $(-\infty, -1] \cup [0, \infty)$

Concave down $[-1, 0]$

Inflection point $(-1, 0)$ & $(0, 1)$

for the graph $y = \sqrt[3]{x^3+1}$

- x-intercept $\longrightarrow \boxed{y=0} \longrightarrow \boxed{x=-1}$ $\in D$
- y-intercept $\longrightarrow \boxed{x=0} \longrightarrow \boxed{y=1}$ $\in D$

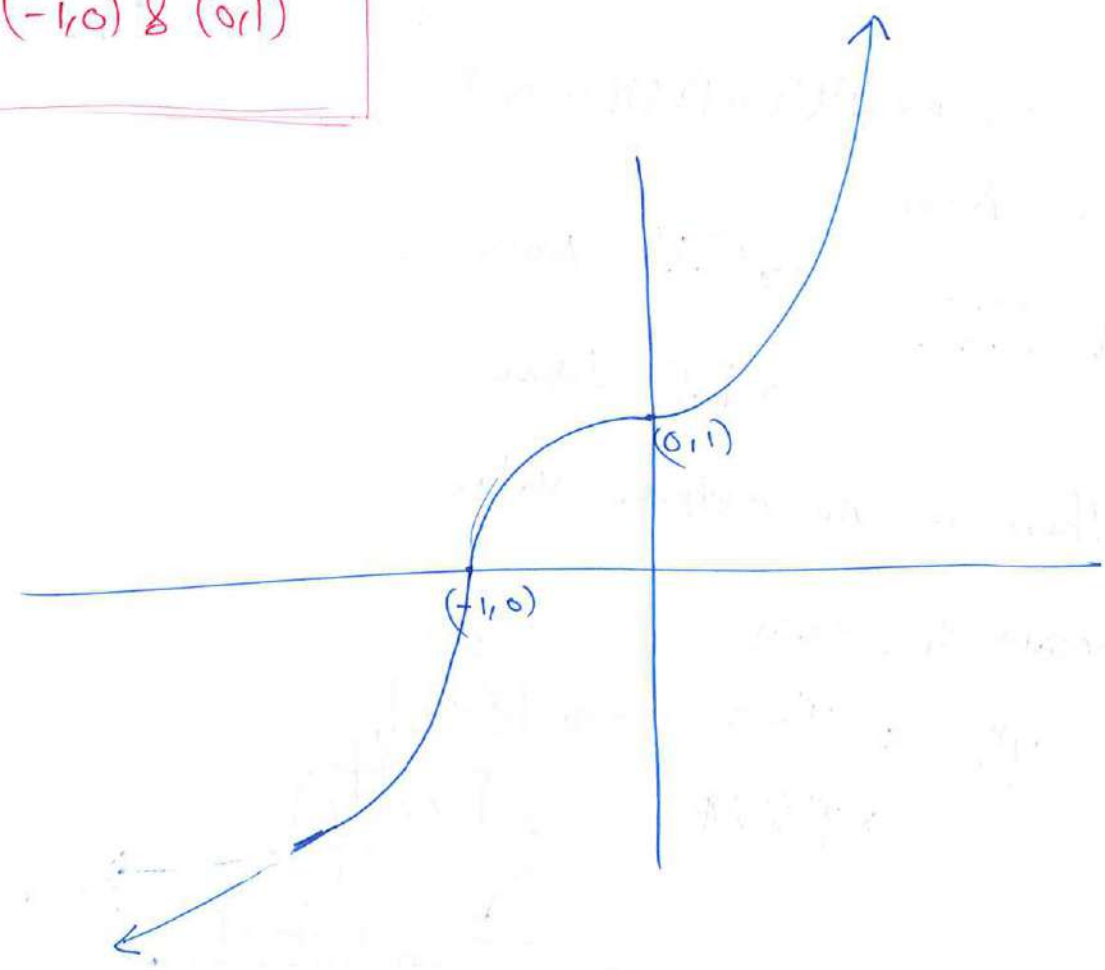
$D = (-\infty, \infty)$

x-intercept $(-1, 0)$

y-intercept $(0, 1)$

~~Max~~

Inflection point $(-1, 0)$ & $(0, 1)$



(f) $y = \dfrac{x}{x^2-1}$ $\quad y' = \dfrac{-1-x^2}{(x^2-1)^2}$ $\quad y'' = \dfrac{2x(x^2+3)}{(x^2-1)^3}$

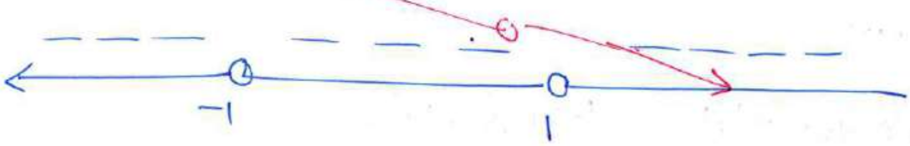
* Domain $\mathbb{R} \setminus \{1,-1\}$

* C.V $\longrightarrow y' = 0 \longrightarrow$ None

$\quad\quad \longrightarrow y'$ DNE $\longrightarrow \boxed{x = \pm 1}$ $\notin D$

There is no critical point

* Inc / Dec.



Dec : $(-\infty,-1) \cup (-1,1) \cup (1,\infty)$

Inc:- None

* E.V $\longrightarrow$ C.V None

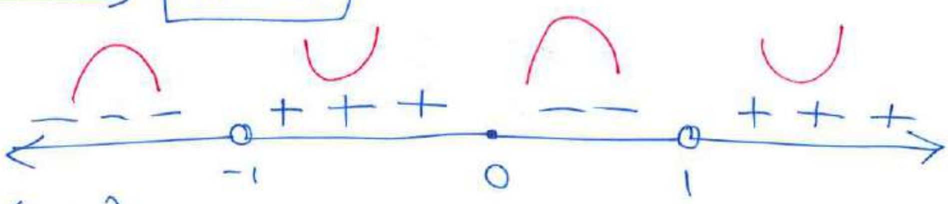
$\quad\quad \longrightarrow$ E.P None

There is no extreme value

Concave up / down

$y'' \longrightarrow y'' = 0 \longrightarrow \boxed{x = 0}$ $\in D$

$\quad\quad \longrightarrow y''$ DNE $\longrightarrow \boxed{x = \pm 1}$ $\notin D$



Concave up $(-1,0] \cup (1,\infty)$

Concave down $(-\infty,-1) \cup [0,1)$ $\quad$ Inflection point $(0,0)$

for the graph $y = \dfrac{x}{x^2-1}$

- x-intercept $\longrightarrow \boxed{y=0} \xrightarrow{6D} \boxed{x=0}$
- y-intercept $\longrightarrow \boxed{x=0} \xrightarrow{6D} \boxed{y=0}$

- H. Asy $\longrightarrow \boxed{y=0}$
- O. Asy $\longrightarrow$ None
- V. Asy $\longrightarrow$ Check at $x=1$, $x=-1$

$$\lim_{x \to 1^+} \frac{x}{x^2-1} = \frac{1}{\text{Small}+} = \infty \qquad \lim_{x \to -1^+} \frac{x}{x^2-1} = \infty$$

$$\lim_{x \to 1^-} \frac{x}{x^2-1} = -\infty \qquad \lim_{x \to -1^-} \frac{x}{x^2-1} = -\infty$$

Then $\boxed{x=1}$, $\boxed{x=-1}$ V. Asy.

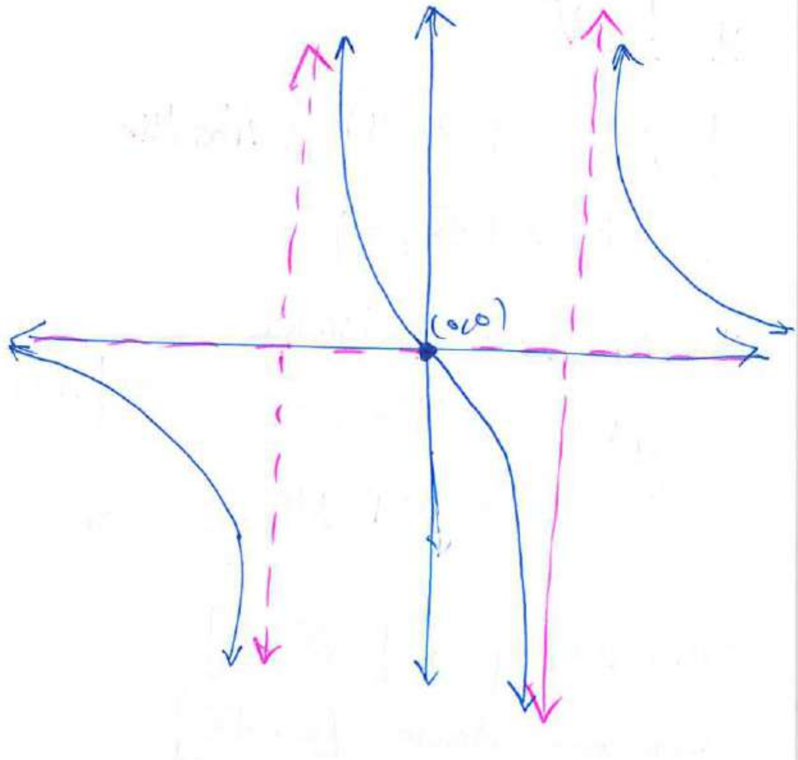
$x \neq 1 \quad x \neq -1$

x-intercept = y-intercept = $(0,0)$

$y=0$ H. Asy

$\left.\begin{array}{l} x=1 \\ x=-1 \end{array}\right\}$ V. Asy

$(0,0)$ Inflection point



(9) $y = x\sqrt{8-x^2}$ $\quad y' = \dfrac{-2x^2+8}{\sqrt{8-x^2}}$ $\quad y'' = \dfrac{2x(x^2-12)}{(8-x^2)^{\frac{3}{2}}}$

* Domain : $\boxed{-\sqrt{8} \leq x \leq \sqrt{8}}$

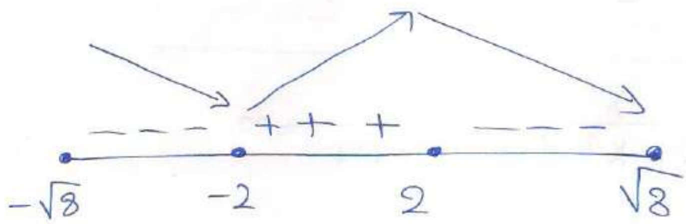
* C.V

$y' = 0 \longrightarrow \overset{\in D}{\boxed{x = \pm 2}}$

y' DNE $\longrightarrow \overset{\in D}{\boxed{x = \pm\sqrt{8}}}$

The critical point = $(2, 4), (-2, 4)$.

* Inc/Dec



y is increasing: $[-2, 2]$

y is decreasing: $[-\sqrt{8}, -2] \cup [2, \sqrt{8}]$

* E.V

L.Min = $(-2, -4)$ & Abs Min

L.Min = $(\sqrt{8}, 0)$

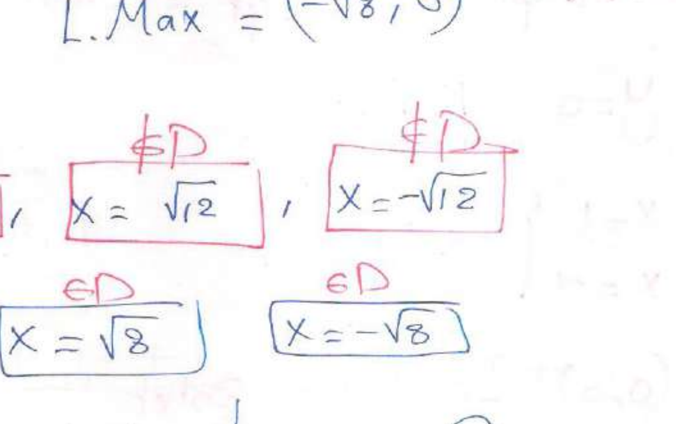
L.Max = $(2, 4)$ & Abs Max

L.Max = $(-\sqrt{8}, 0)$

* Concave up / down

$y'' = 0 \longrightarrow \overset{\in D}{\boxed{x = 0}}, \overset{\notin D}{\boxed{x = \sqrt{12}}}, \overset{\notin D}{\boxed{x = -\sqrt{12}}}$

y'' DNE $\longrightarrow \overset{\in D}{\boxed{x = \sqrt{8}}} \quad \overset{\in D}{\boxed{x = -\sqrt{8}}}$



Concave up = $[-\sqrt{8}, 0]$

Concave down $[0, \sqrt{8}]$

Inflection point $(0, 0)$

for the graph

- x-intercept $\longrightarrow$ $y=0$ $\longrightarrow$ $(0,0)$ $(\sqrt{8},0)$ $(-\sqrt{8},0)$

- y-intercept $\longrightarrow$ $x=0$ $\longrightarrow$ $(0,0)$

$-\sqrt{8} \leq x \leq \sqrt{8}$

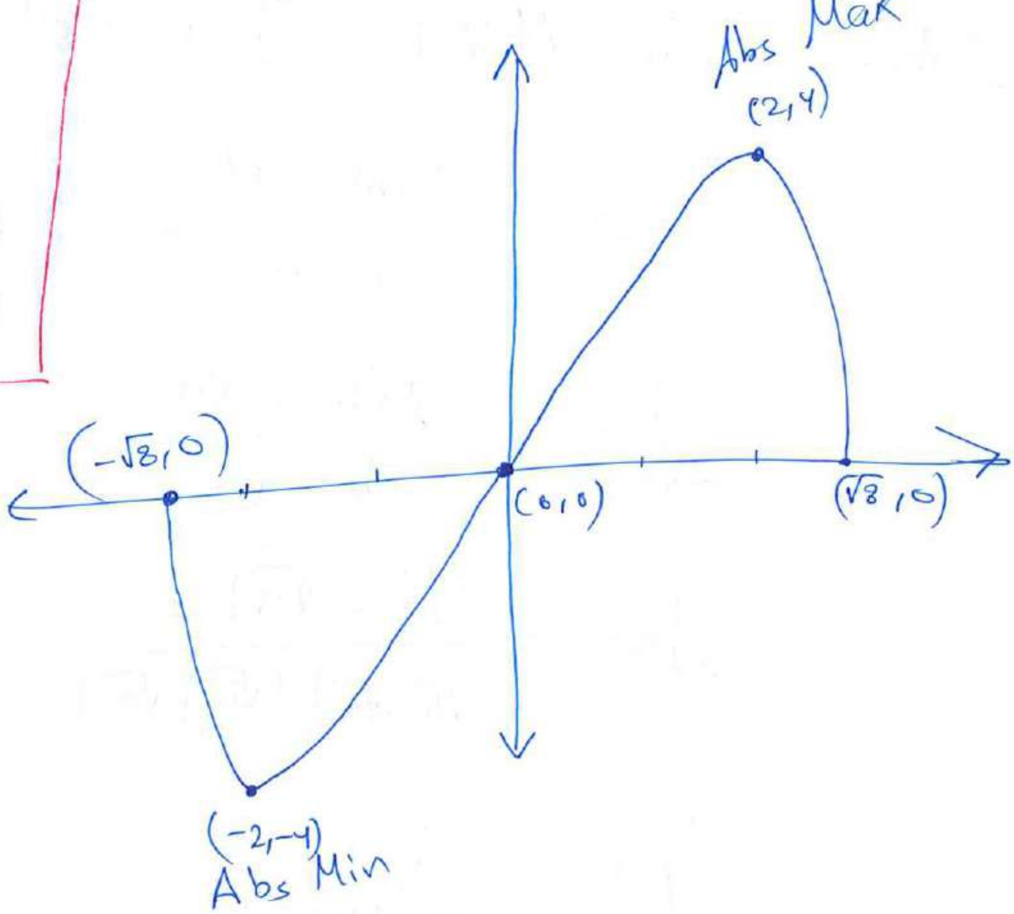
x-intercept $(-\sqrt{8},0)$, $(0,0)$, $(\sqrt{8},0)$

y-intercept $(0,0)$

Max $(\sqrt{8},0)$ $(2,4)$

Min $(-2,-4)$ $(\sqrt{8},0)$

Inflection point $(0,0)$



Q_2 By Using M.V.T Find c ??

$$f(x) = \sqrt{x}$$

- $f(x)$ is cont on $[a,b]$, since $a > 0$
- $f'(x) = \frac{1}{2\sqrt{x}}$ $f(x)$ is diff on (a,b)

Then $\Rightarrow$ by M.V.T $\exists$ at least $c \in (a,b)$ s.th

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\frac{1}{2\sqrt{c}} = \frac{\sqrt{b} - \sqrt{a}}{b - a}$$

$$\frac{1}{2\sqrt{c}} = \frac{(\sqrt{b} - \sqrt{a})}{(\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a})}$$

$$\frac{1}{2\sqrt{c}} = \frac{1}{\sqrt{b} + \sqrt{a}}$$

$$\boxed{c = \left(\frac{\sqrt{b} + \sqrt{a}}{2}\right)^2}$$

$\boxed{Q_3}$ Find a, m & $b = ??$

since $f(x)$ satisfy the hypotheses of the M.V.T

$$f(x) = \begin{cases} 3 & x = 0 \\ -x^2 + 3x + a & 0 < x < 1 \\ mx + b & 1 \leq x \leq 2 \end{cases}$$

- Since $f(x)$ is cont on $[0, 2]$

$\longrightarrow$ f is cont at $x=0$

$$f(0) = \lim_{x \to 0^+} f(x)$$

$$3 = \lim_{x \to 0^+} -x^2 + 3x + a$$

$$\boxed{3 = a}$$

$\longrightarrow$ f is cont at $x=1$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$$

$$\lim_{x \to 1^+} mx + b = \lim_{x \to 1^-} -x^2 + 3x + 3$$

$$\boxed{m + b = 5}$$

- Since $f(x)$ is diff on $(0,2)$

$\longrightarrow$ f is diff at $x=1$

$$f'(x) = \begin{cases} -2x + 3 & 0 < x < 1 \\ m & 1 < x < 2 \end{cases}$$

$$f'(1)^+ = f'(1)^-$$

$$\boxed{m = 1}$$

Then $\boxed{a = 3}$, $\boxed{m = 1}$, $\boxed{b = 4}$