

Ch. 5 :- Integration

1

(a) $\int \sin(5x) dx$

$$= \frac{-\cos(5x)}{5} + C = \boxed{\frac{-1}{5} \cos(5x) + C}$$

(b) $\int \tan^2 x dx$

$$= \int (\sec^2 x - 1) dx$$

$$= \boxed{\tan x - x + C}$$

(c) $\int (1 + \cot^2 \theta) d\theta$

$$\int (1 + \csc^2 \theta - 1) d\theta = \int \csc^2 \theta d\theta$$

$$= \boxed{-\cot \theta + C}$$

(d) $\int \frac{\csc \theta d\theta}{\csc \theta - \sin \theta}$

$$= \int \frac{\frac{1}{\sin \theta}}{\frac{1}{\sin \theta} - \sin \theta} d\theta = \int \frac{\frac{1}{\sin \theta}}{\frac{1 - \sin^2 \theta}{\sin \theta}} d\theta$$

$$= \int \frac{1}{\cos^2 \theta} d\theta = \int \sec^2 \theta d\theta$$

$$= \boxed{\tan \theta + C}$$

2 Find the derivative of the following functions:-

$$(a) \quad y = \int_1^x \frac{dt}{t}$$

$$y' = \frac{1}{x} - 1 \cdot (0) = \frac{1}{x} \cdot$$

$$(b) \quad y = \int_0^{\sqrt{x}} \cos t \, dt$$

$$y' = \cos(\sqrt{x}) \cdot \left(\frac{1}{2\sqrt{x}}\right) \cdot$$

$$= \frac{\cos \sqrt{x}}{2\sqrt{x}}$$

$$(c) \quad y = \int_{\tan x}^0 \frac{dt}{1+t^2}$$

$$y = - \int_0^{\tan x} \frac{dt}{1+t^2}$$

$$y' = - \frac{1}{1+\tan^2 x} \cdot \sec^2 x$$

Note: $1 + \tan^2 x = \sec^2 x$
 $1 + \cot^2 x = \csc^2 x$

$$= \frac{-\sec^2 x}{\sec^2 x} = \boxed{-1}$$

3] Find the linearization of $g(x) = 3 + \int_1^{x^2} \sec(t-1) dt$
at $x = -1$

The linearization of $f(x)$ at $x = a$

$$L(x) = f(a) + f'(a)(x-a)$$

$$g(-1) = 3 + \int_1^1 \sec(t-1) dt = 3$$

$$g'(x) = 0 + \sec(x^2-1) \cdot 2x = 2x \sec(x^2-1)$$

$$g'(-1) = -2 \sec(0) = -2$$

→ Then the linearization of $g(x)$ at $x = -1$

$$L(x) = 3 + -2(x+1)$$

$$L(x) = 3 - 2x - 2 = 1 - 2x$$

Extra question Estimate the $g(x)$ value if $x = 0.9$

$$L(0.9) \approx g(0.9)$$

$$L(0.9) = 1 - 2(0.9) = -0.8$$

4] Solve the following definite Integrals:

$$\begin{aligned}
 \text{(a)} \quad & \int_1^{\sqrt{2}} \frac{s^2 + \sqrt{s}}{s^2} ds \\
 & \int_1^{\sqrt{2}} 1 + \frac{\sqrt{s}}{s^2} ds = \int_1^{\sqrt{2}} 1 + s^{-\frac{3}{2}} ds \\
 & = s + -2 \cdot s^{-\frac{1}{2}} \Big|_1^{\sqrt{2}} \\
 & = \sqrt{2} - 2 (\sqrt{2})^{-\frac{1}{2}} - [1 - 2] \\
 & = \sqrt{2} - 2 (2)^{-\frac{1}{4}} + 1 \\
 & = \sqrt{2} - 2^{3/4} + 1 \\
 & = \boxed{\sqrt{2} - \sqrt[4]{8} + 1}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & \int_0^{\pi/6} (\sec x + \tan x)^2 dx \\
 & = \int_0^{\pi/6} \sec^2 x + 2 \sec x \tan x + \tan^2 x dx \\
 & = \int_0^{\pi/6} (\sec^2 x + 2 \sec x \tan x + \sec^2 x - 1) dx \\
 & = \int_0^{\pi/6} (2 \sec^2 x + 2 \sec x \tan x - 1) dx \\
 & = 2 \tan x + 2 \sec x - x \Big|_0^{\pi/6} \\
 & = 2 \cdot \left(\frac{1}{\sqrt{3}}\right) + 2 \left(\frac{2}{\sqrt{3}}\right) - \frac{\pi}{6} - (2(0) + 2(1) - 0) \\
 & = \frac{2}{\sqrt{3}} + \frac{4}{\sqrt{3}} - \frac{\pi}{6} - 2 = \boxed{\frac{6}{\sqrt{3}} - \frac{\pi}{6} - 2}
 \end{aligned}$$

$$\boxed{C} \int_0^{\pi} \cos x + |\cos x| dx$$

$$|\cos x| = \begin{cases} \cos x & 0 \leq x \leq \frac{\pi}{2} \\ -\cos x & \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

$$\int_0^{\pi/2} 2 \cos x dx + \int_{\pi/2}^{\pi} (\cos x - \cos x) dx$$

$$\int_0^{\pi/2} 2 \cos x dx + \int_{\pi/2}^{\pi} 0 dx$$

$$= 2 \cdot \sin x \Big|_0^{\pi/2} = 2(1) - 2(0) = 2$$

Q5 Use Substitution to solve the following integral:-

$$\text{(a)} \int \frac{dx}{\sqrt{x}(1+\sqrt{x})^2}$$

Assume $U = 1 + \sqrt{x}$
 $dU = \frac{1}{2\sqrt{x}} dx$

$$\int \frac{2\sqrt{x} dU}{\sqrt{x} (U)^2}$$

$$\frac{2 \cdot U^{-1}}{-1} + C = \frac{-2}{U} + C$$

last step

$$= \boxed{\frac{-2}{1+\sqrt{x}} + C}$$

$$\text{b) } \int \frac{\sec z \tan z}{\sqrt{\sec z}} dz$$

let $w = \sec z$
 $dw = \sec z \tan z dz$

$$\int \frac{\cancel{\sec z} \cancel{\tan z}}{\sqrt{w}} \frac{dw}{\cancel{\sec z} \cancel{\tan z}}$$

$$\int w^{-1/2} dw = 2w^{1/2} + C$$

$$= 2\sqrt{\sec z} + C$$

$$\text{c) } \int \sqrt{\frac{x-1}{x^5}} dx$$

$$\int \sqrt{\frac{x-1}{x}} \cdot \frac{1}{x^4} dx$$

$$\int \sqrt{\frac{x-1}{x}} \cdot \frac{dx}{x^2}$$

let $u = \frac{x-1}{x}$

$$u = 1 - \frac{1}{x}$$

$$du = (0 - -x^{-2}) dx$$

$$du = \frac{1}{x^2} dx$$

$$\int \sqrt{u} \frac{x^2 du}{x^2}$$

$$\int \sqrt{u} du$$

$$= \frac{2}{3} u^{3/2} + C$$

$$= \frac{2}{3} \left(\frac{x-1}{x} \right)^{3/2} + C$$

$$\boxed{d} \int x^3 \sqrt{x^2 + 1} dx$$

$$\text{let } z = x^2 + 1$$

$$\int x^3 \sqrt{z} \frac{dz}{2x}$$

$$dz = 2x dx$$

$$\frac{1}{2} \int x^2 \sqrt{z} dz$$

$$\frac{1}{2} \int (z - 1) z^{1/2} dz$$

$$\frac{1}{2} \int z^{3/2} - z^{1/2} dz$$

$$\frac{1}{2} \left[\frac{2}{5} z^{5/2} - \frac{2}{3} z^{3/2} \right] + C$$

$$\frac{1}{5} z^{5/2} - \frac{1}{3} z^{3/2} + C$$

$$= \frac{1}{5} (x^2 + 1)^{5/2} - \frac{1}{3} (x^2 + 1)^{3/2} + C$$

Q6 Find the area enclosed between the given functions [in the x-y plane]

مساحة المجال المحاطة بين الرسمين باستخدام التكامل المركب Area

$$\text{Area} = \left| \int_a^b f(x) - g(x) dx \right|$$

since a, b are the intersection points

$$\text{Area} = \left| \int_c^d f(y) - g(y) dy \right|$$

c, d are the intersection points

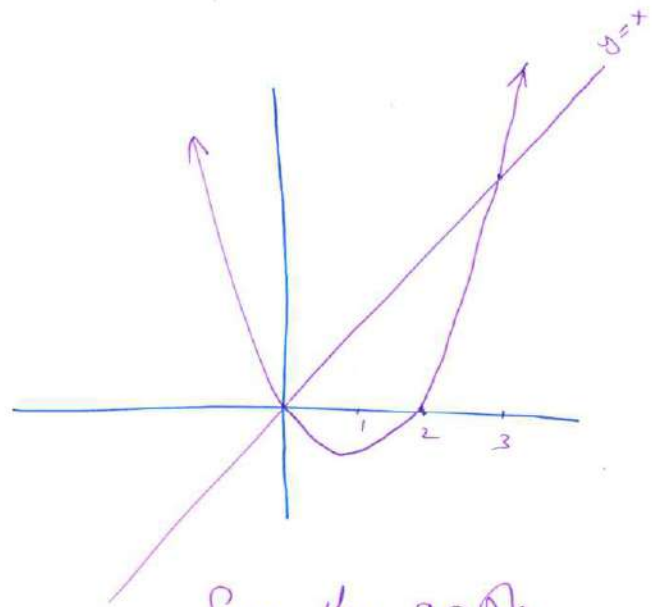
(a) $y = x^2 - 2x$, $y = x$

Intersection point :-

$$x^2 - 2x = x$$

$$x^2 - 3x = 0$$

$$\boxed{x=0} \quad \boxed{x=3}$$



from the graph

$$x > x^2 - 2x \text{ in } [0, 3]$$

$$A = \int_0^3 x - (x^2 - 2x) dx$$

$$= \int_0^3 x - x^2 + 2x dx = \int_0^3 3x - x^2 dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3$$

$$= \frac{27}{2} - \frac{27}{3} - 0$$

$$= \frac{27}{2} - 9 = \frac{9}{2}$$

$$\textcircled{b} \quad y = x^2 \quad , \quad y = -x^2 + 4x$$

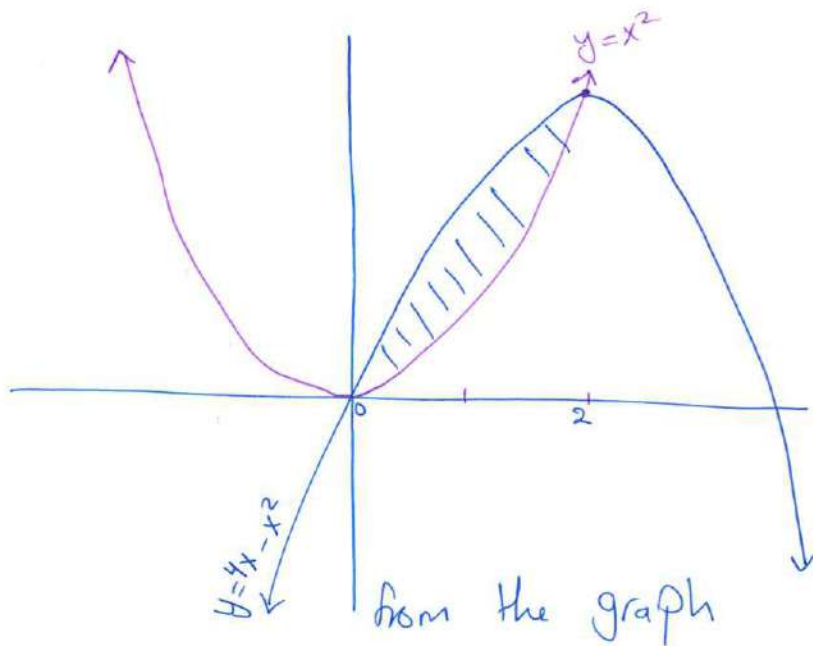
Intersection point:

$$x^2 = -x^2 + 4x$$

$$2x^2 - 4x = 0$$

$$2x [x - 2] = 0$$

$$\boxed{x=0} \quad \boxed{x=2}$$



$$4x - x^2 > x^2$$

$$A = \int_0^2 4x - x^2 - x^2 \, dx$$

$$\int_0^2 4x - 2x^2 \, dx$$

$$\left[\frac{4x^2}{2} - \frac{2x^3}{3} \right]_0^2 = 8 - \frac{16}{3} = \boxed{\frac{8}{3}}$$

$$\textcircled{c} \quad x = y^2$$

$$x = 3 - 2y^2$$

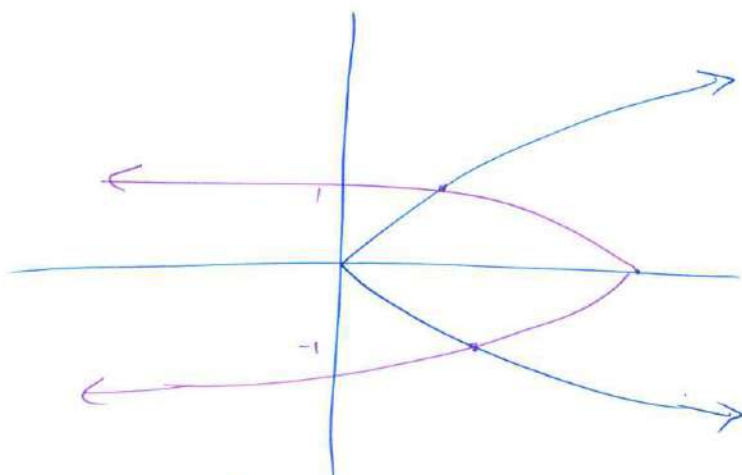
Intersection point

$$y^2 = 3 - 2y^2$$

$$3y^2 = 3$$

$$y^2 = 1$$

$$\boxed{y = \pm 1}$$



from the graph.

$$\boxed{3 - 2y^2 > y^2} \quad \text{in the interval} \quad y = [-1, 1]$$

$$\begin{aligned} A &= \int_{-1}^1 (3 - 2y^2 - y^2) dy \\ &= \int_{-1}^1 (3 - 3y^2) dy = \left[3y - \frac{3y^3}{3} \right]_{-1}^1 \\ &= 3 - 1 - [-3 + 1] = \boxed{4} \end{aligned}$$

$$\textcircled{a} \quad X = y^3 - y^2$$

$$X = 2y$$

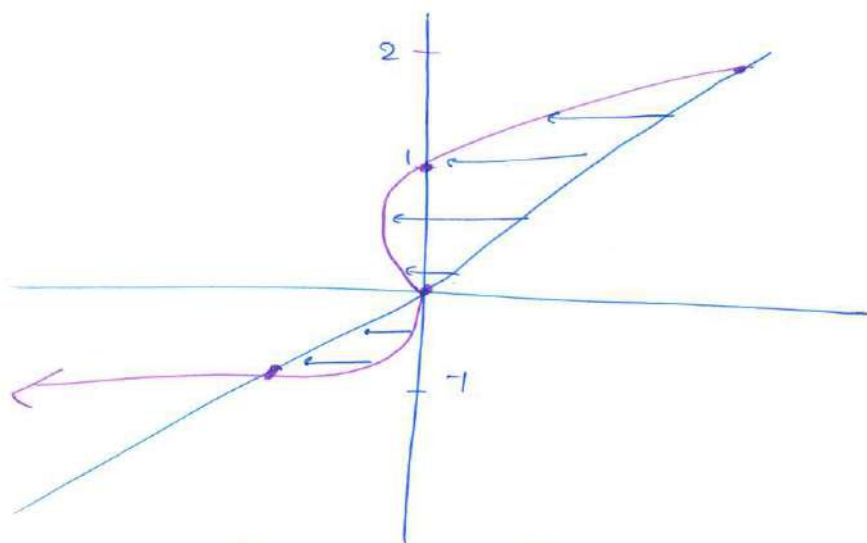
Intersection point

$$y^3 - y^2 = 2y$$

$$y^3 - y^2 - 2y = 0$$

$$y(y^2 - y - 2) = 0$$

$$\boxed{y=0} \quad \boxed{y=-1} \quad \boxed{y=2}$$



from the graph

$$2y > y^3 - y^2$$

$$y \in [0, 2]$$

$$y^3 - y^2 > 2y$$

$$y \in [-1, 0]$$

$$A = \int_{-1}^0 () dy + \int_0^2 () dy$$

$$A = \int_{-1}^0 (y^3 - y^2 - 2y) dy + \int_0^2 (2y - y^3 + y^2) dy$$

$$= \left[\frac{y^4}{4} - \frac{y^3}{3} - \frac{2y^2}{2} \right]_{-1}^0 + \left[\frac{2y^2}{2} - \frac{y^4}{4} + \frac{y^3}{3} \right]_0^2$$

$$= 0 - \left[\frac{1}{4} + \frac{1}{3} - \frac{1}{1} \right] + \left[4 - 4 + \frac{8}{3} \right]$$

$$\frac{5}{12} + \frac{8}{3} = \boxed{\frac{37}{12}}$$