6.1: ⇒ Volume Using Cross-Sections

* The Disk Method

- Graph
- Volume
  - → About x-axis / About Horizantal line: $$V = \int_{\square}^{\square} A(x)\, dx$$
  - → About y-axis / About vertical line: $$V = \int_{\square}^{\square} A(y)\, dy$$

- $R(x)$ / $R(y)$ ⟶ $A(x) = \pi R^2(x)$ (Disk)

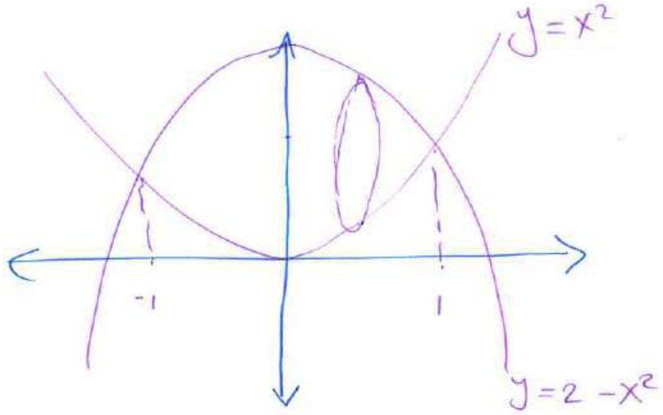
* The Washer Method

we use this method if the solid of revolution has a hole in it.

- Graph
- Volume
  - → about x-axis / about horizantal line: $$V = \int_{\square}^{\square} A(x)\, dx = \int_{\square}^{\square} \pi \left[R^2(x) - r^2(x)\right] d$$
  - → about y-axis / about Vertical line: $$V = \int_{\square}^{\square} A(y)\, dy = \int_{\square}^{\square} \pi \left[R^2(y) - r^2(y)\right] dy$$

Note: Disk & Washer Method رسم خط يقطع منطقة التكامل وعمودي على محور الدوران

Q2 The Solid lies between planes perpendicular to the x_axis at $x=-1$ & $x=1$. The cross sections perpendicular to the x_axis are circular disk whose diameters run from the parabola $y=x^2$ to the parabola $y=2-x^2$



Solution:-

Cross section (Disk) $\perp$ x_axis

$$V = \int_{\square}^{\square} A(x)\, dx$$

$$V = \int_{-1}^{1} \pi R^2(x)\, dx$$

$$D = 2 - x^2 - x^2$$

$$D = 2 - 2x^2$$

$$R = \frac{1}{2} D = 1 - x^2$$

$$R^2 = (1-x^2)^2 = 1 - 2x^2 + x^4$$

$$= \int_{-1}^{1} \pi \left[1 - 2x^2 + x^4\right] dx$$

$$= \pi \left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_{-1}^{1}$$

$$= \pi \left[1 - \frac{2}{3} + \frac{1}{5} + 1 - \frac{2}{3} + \frac{1}{5}\right]$$

$$= \boxed{\frac{16\pi}{15}}$$

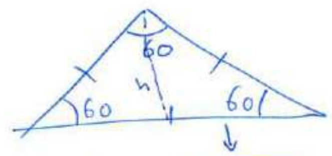
Q5 (a) Equilateral triangle :- مثلث متساوي الاضلاع

Cross section [equilateral triangle] $\perp$ X-axis

$$A = \int_{\boxed{0}}^{\boxed{\pi}} A(x)\, dx$$

$A(x) = \text{Area} = \frac{1}{2}(\text{side})(h)$

$= \frac{1}{2}(\text{side})(\text{side})(\sin 60)$



$\text{side} = 2\sqrt{\sin x}$

$$= \int_0^{\pi} \frac{1}{2} (\text{side})^2 (\sin 60)\, dx$$

$$= \int_0^{\pi} \frac{1}{2} \cdot 4\sin x \cdot \left(\frac{\sqrt{3}}{2}\right) dx$$

$$= \int_0^{\pi} \sqrt{3} \sin x\, dx = -\sqrt{3}\left[\cos x\right]_0^{\pi} = -\sqrt{3}\,[-2] = \boxed{2\sqrt{3}}$$

(b) Cross section (Square) $\perp$ X-axis

$$V = \int_{\boxed{0}}^{\boxed{\pi}} A(x)\, dx = \int_0^{\pi} (\text{side})^2\, dx$$

$$= \int_0^{\pi} 4\sin x\, dx = -4\cos x \Big]_0^{\pi}$$

$$= -4 \cdot [-2] = \boxed{8}$$

Q9 The solid lies between planes perpendicular to the y-axis at $y=0$ & $y=2$, The cross section perpendicular to the y-axis are circular disk with diameter running from the y-axis to the parabola $x = \sqrt{5}\, y^2$

Solution

Cross section (Disk) $\perp$ y-axis

$$V = \int_{\boxed{0}}^{\boxed{2}} A(y)\, dy$$

$D = \sqrt{5}\, y^2 - 0$

$D = \sqrt{5}\, y^2$

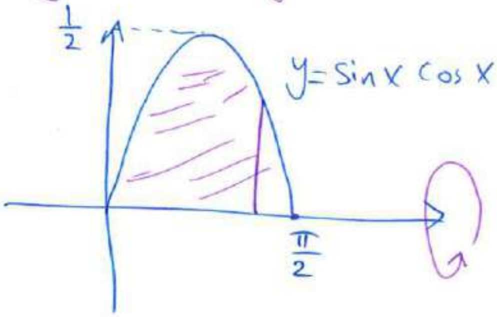
$R = \frac{1}{2} D = \frac{\sqrt{5}}{2} y^2$

$R^2(y) = \frac{5}{4} y^4$

$$= \int_0^2 \pi R^2(y)\, dy = \int_0^2 \pi \left(\frac{5}{4} y^4\right) dy$$

$$= \frac{5\pi}{4} \left.\frac{y^5}{5}\right]_0^2 = \frac{\pi}{4}[32] = \boxed{8\pi}$$

Q18 Find Volume of the solid generated by revolving the region about the x-axis



$$V = \int_{\boxed{0}}^{\boxed{\pi/2}} A(x)\, dx$$

$$= \int_0^{\pi/2} \pi (\sin x \cos x)^2\, dx$$

$$= \pi \int_0^{\pi/2} \left(\frac{\sin 2x}{2}\right)^2 dx$$

$$= \frac{\pi}{4} \int_0^{\pi/2} \sin^2(2x)\, dx$$

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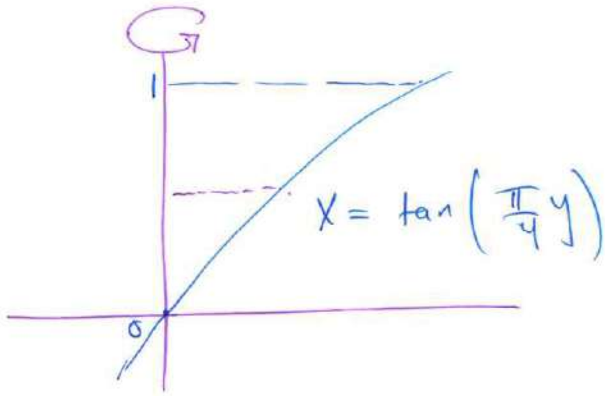
$$= \frac{\pi}{4} \int_0^{\pi/2} \sin^2(2x) \, dx$$

$$= \frac{\pi}{4} \int_0^{\pi/2} \frac{1 - \cos(4x)}{2} \, dx$$

$$= \frac{\pi}{4} \left[\frac{1}{2}x - \frac{\sin(4x)}{8} \right]_0^{\pi/2}$$

$$= \frac{\pi}{4} \left[\frac{\pi}{4} - 0 - 0 \right] = \boxed{\frac{\pi^2}{16}}$$

$\boxed{Q_{17}}$ About the y-axis



$$V = \int_{\boxed{0}}^{\boxed{1}} A(y) \, dy$$

$$R(y) = \tan\left(\frac{\pi}{4} y\right) - 0$$

$$R^2(y) = \tan^2\left(\frac{\pi}{4} y\right)$$

$$= \int_0^1 \pi \tan^2\left(\frac{\pi}{4} y\right) dy$$

$$= \pi \int_0^1 \sec^2\left(\frac{\pi}{4} y\right) - 1 \, dy$$

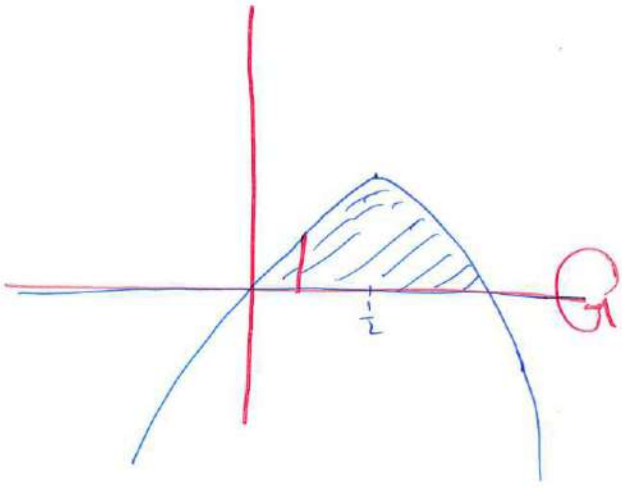
$$= \pi \left[\frac{\tan(\pi/4\, y)}{\pi/4} - y \right]_0^1 = \pi \left[\frac{4}{\pi} \tan\left(\frac{\pi}{4}\right) - 1 \right]$$

$$= \pi \left[\frac{4}{\pi} - 1 \right]$$

$$= \boxed{4 - \pi}$$

$\boxed{Q_{22}}$ Find the Volumes of the solids generated by revolving the region bounded by the line and curves in Exercises 19-24 about the x-axis

22. $y = x - x^2$, $y = 0$



$$V = \int_{\boxed{0}}^{\boxed{1}} A(x)\, dx$$

$$= \int_0^1 \pi\, R^2(x)\, dx$$

$$= \pi \int_0^1 x^2 - 2x^3 + x^4\, dx$$

$$= \pi \left[\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5} \right]_0^1$$

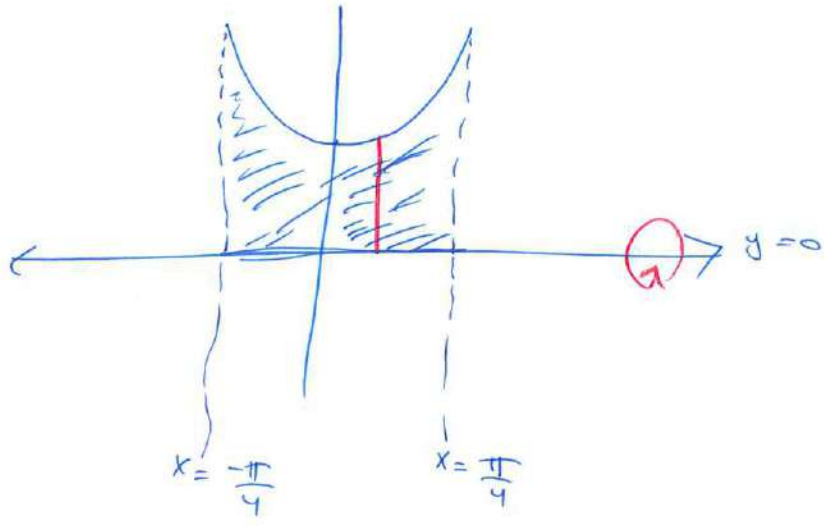
$$= \pi \left[\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right] = \boxed{\frac{\pi}{30}}$$

$R(x) = x - x^2 - 0$

$R^2(x) = x^2 - 2x^3 + x^4$

$\boxed{Q_{21}}$ $y = \sec x$, $y = 0$, $x = \frac{-\pi}{4}$, $x = \frac{\pi}{4}$

about x-axis



$$V = \int_{\boxed{-\frac{\pi}{4}}}^{\boxed{\frac{\pi}{4}}} A(x) \, dx$$

$$= \int_{-\pi/4}^{\pi/4} \pi R^2(x) \, dx$$

$R(x) = \sec x - 0$

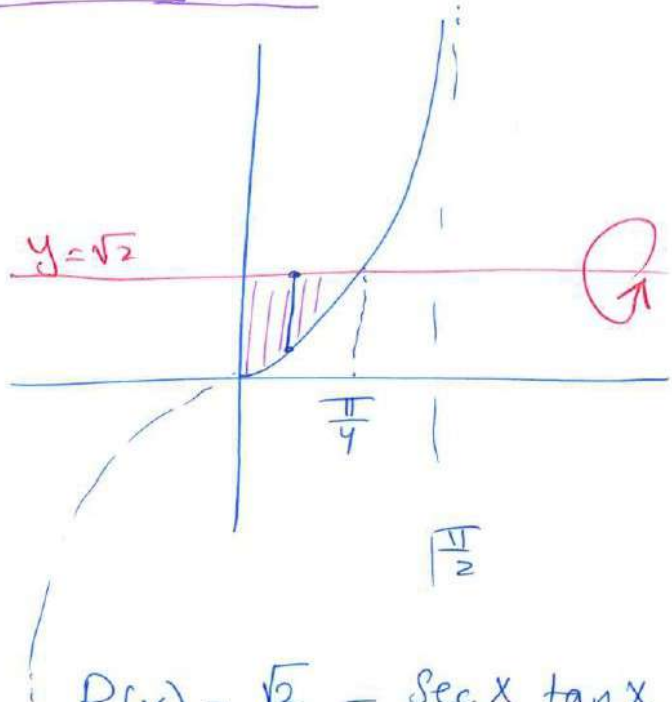
$R^2(x) = \sec^2 x$

$$= \int_{-\pi/4}^{\pi/4} \pi \sec^2 x \, dx$$

$$= 2\pi \int_{0}^{\pi/4} \sec^2 x \, dx = 2\pi \Big[\tan x\Big]_0^{\pi/4}$$

$$= 2\pi \, [1 - 0] = \boxed{2\pi}$$

$\boxed{Q_{25}}$ Find the volume of the solid generated by revolving the region in the First quadrant bounded ~~by~~ above by $y=\sqrt{2}$, below by $y = \sec x \tan x$, and on the left by y-axis about the line $y=\sqrt{2}$



$$V = \int_{\boxed{0}}^{\boxed{\pi/4}} A(x)\, dx$$

$$V = \int_0^{\pi/4} \pi R^2(x)\, dx$$

$$R(x) = \sqrt{2} - \sec x \tan x$$

$$R^2(x) = 2 - 2\sqrt{2} \sec x \tan x + \sec^2 x \tan^2 x$$

$$V = \pi \int_0^{\pi/4} 2 - \sqrt{8} \sec x \tan x$$

$$+ \pi \int_0^{\pi/4} \sec^2 x \tan^2 x\, dx$$

$$V = \pi \Big[2x - \sqrt{8} \sec x \Big]_0^{\pi/4} + \pi \int_{\boxed{0}}^{\boxed{1}} \cancel{\sec^2 x}\, U^2 \frac{dU}{\cancel{\sec^2 x}}$$

let $U = \tan x$

$dU = \sec^2 x\, dx$

$x = 0 \to U = 0$

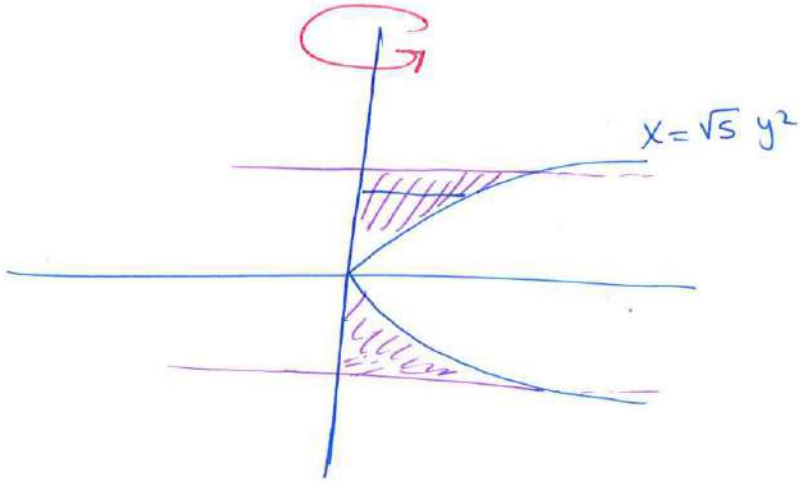
$x = \frac{\pi}{4} \to U = 1$

$$= \pi \left[\frac{\pi}{2} - 4 + \sqrt{8} \right] + \pi \left[\frac{U^3}{3} \right]_0^1$$

$$= \pi \left[\frac{\pi}{2} - 4 + \sqrt{8} + \frac{1}{3} \right] = \pi \left[\frac{\pi}{2} + 2\sqrt{2} - \frac{11}{3} \right]$$

$\boxed{Q_{27}}$ Find the volume of the solid generated by revolving the regions bounded by the line and curve about y-axis $\boxed{x = \sqrt{5}\, y^2}$ $\boxed{x=0}$ $\boxed{y=-1}$, $\boxed{y=1}$

$$V = \int_{\boxed{-1}}^{\boxed{1}} A(y)\, dy$$



$$V = \int_{-1}^{1} \pi\, R^2(y)\, dy$$

$$= \int_{-1}^{1} \pi \left[\sqrt{5}\, y^2\right]^2 dy$$

for the graph

$x = \sqrt{5}\, y^2 \rightarrow |y| = \frac{1}{\sqrt[4]{5}} \sqrt{x}$

Cross section (Disk Method)

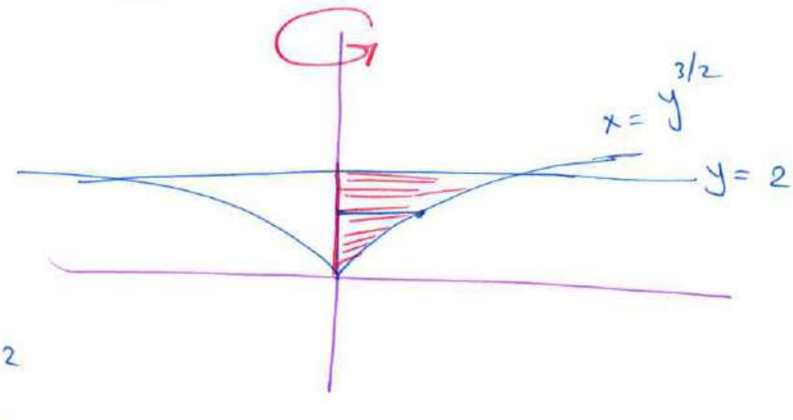
$$= \int_{-1}^{1} 5\pi\, y^4\, dy$$

$$= 5\pi \left.\frac{y^5}{5}\right]_{-1}^{1}$$

$$= \pi \left[1+1\right] = 2\pi$$

$\boxed{Q_{28}}$ $\boxed{x = y^{3/2}}$, $\boxed{x=0}$, $\boxed{y=2}$ about y-axis

$$V = \int_{\boxed{0}}^{\boxed{2}} A(y)\, dy$$

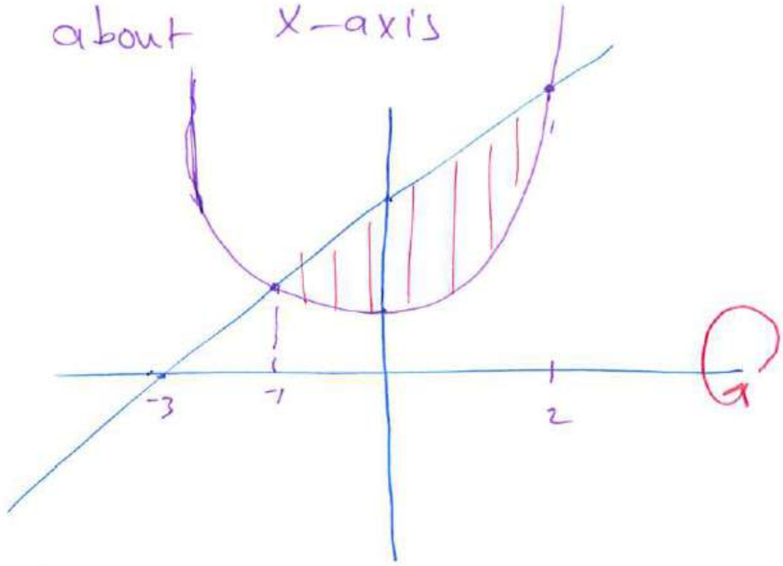


$$V = \int_0^2 \pi\, y^3\, dy = \pi \left.\frac{y^4}{4}\right]_0^2$$

$$= \boxed{4\pi}$$

Q37 Find the volume of the Solid generated by revolving the region bounded by the line and curve

$y = x^2+1, \quad y = x+3$ about x-axis



Using Washer Method:

$$V = \int_{-1}^{2} A(x)\,dx$$

$$= \int_{-1}^{2} \pi \left[R^2(x) - r^2(x)\right] dx$$

$r(x) = x^2+1$

$R(x) = x+3$

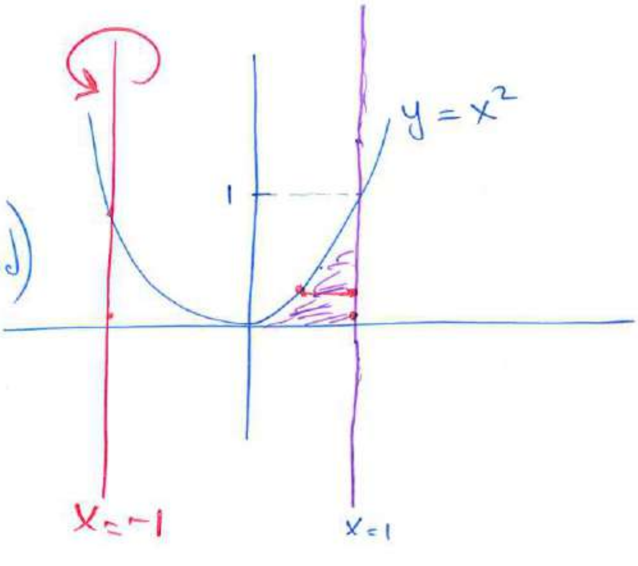
$$= \int_{-1}^{2} \pi \left[x^2+6x+9 - x^4 - 2x^2 - 1\right] dx$$

$$= \pi \int_{-1}^{2} -x^2+6x-x^4+8 \; dx$$

$$\pi \left[\frac{-x^3}{3} + \frac{6x^2}{2} - \frac{x^5}{5} + 8x\right]_{-1}^{2}$$

$$\pi \left[\frac{-8}{3} + 12 - \frac{32}{5} + 16 - \frac{1}{3} - 3 - \frac{1}{5} + 8\right] = \boxed{\frac{117\pi}{5}}$$

$\boxed{Q_{45}}$ The region in the first quadrant bounded above by the curve $y=x^2$, below by the x-axis, on the right by $x=1$ about $\boxed{x=-1}$



Solution (By using washer Method) about Vertical line

$$V=\int_{\boxed{0}}^{\boxed{1}} A(y)\,dy$$

$R(y)=1--1=2$

$r(y)=\sqrt{y}--1$

$r(y)=\sqrt{y}+1$

$$=\int_0^1 \pi\left[R^2(y)-r^2(y)\right]dy$$

$$=\int_0^1 \pi\left[4-\left(\sqrt{y}+1\right)^2\right]dy$$

$$=\pi\int_0^1 4-y-2\sqrt{y}-1\,dy$$

$$=\pi\left[3y-\frac{y^2}{2}-\frac{4}{3}y^{3/2}\right]_0^1$$

$$=\pi\left[3-\frac{1}{2}-\frac{4}{3}\right]=\boxed{\frac{7\pi}{6}}$$

$\boxed{Q_{47}}$ Find the volume of the solid generated by the region bounded by $\boxed{y=\sqrt{x}}$ and the line $\boxed{y=2}$ & $\boxed{x=0}$

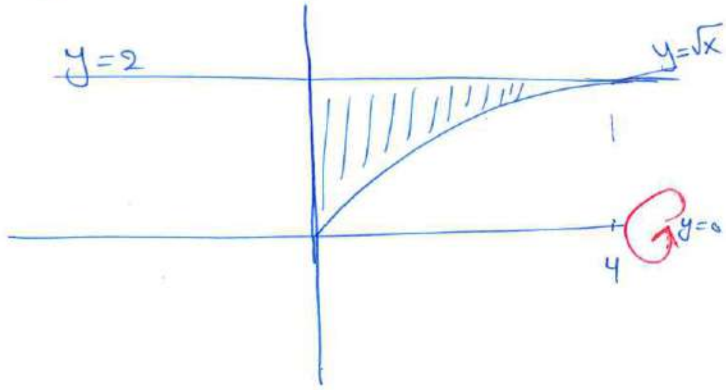
about

(a) the x-axis

(b) the y-axis

(c) the line $y=2$ [horizantal line]

(d) the line $x=4$ [vertical line]



(a) About x-axis

By using washer Method

$$V = \int_{\boxed{0}}^{\boxed{4}} A(x)\, dx$$

$r(x) = \sqrt{x}$

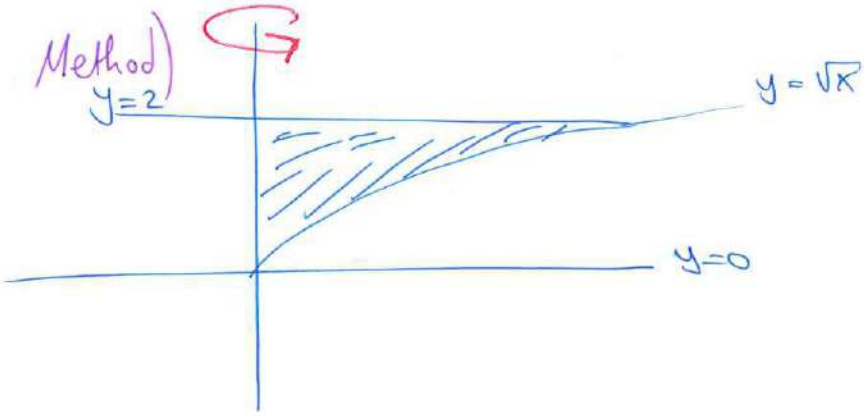
$R(x) = 2$

$$= \int_0^4 \pi \left[R^2(x) - r^2(x)\right] dx$$

$$= \int_0^4 \pi \left[4 - x\right] dx$$

$$= \pi \left[4x - \frac{x^2}{2}\right]_0^4 = 8\pi$$

(b) About y-axis (Disk Method)



$$V = \int_{\boxed{0}}^{\boxed{2}} A(y)\, dy$$

$$= \int_0^2 \pi y^4 \, dy$$

$R(y) = y^2 - 0$

$R(y) = y^2$

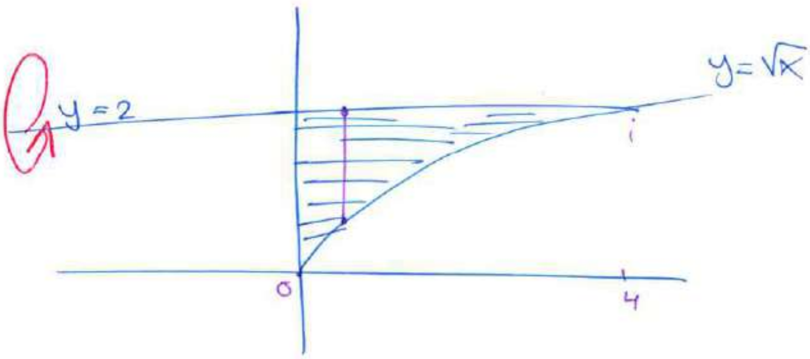
$R^2(y) = y^4$

$$= \pi \left. \frac{y^5}{5} \right]_0^2$$

$$= \boxed{\frac{32}{5}\pi}$$

(c) About $y=2$

(Disk Method)



$$V = \int_{\boxed{0}}^{\boxed{4}} A(x)\, dx$$

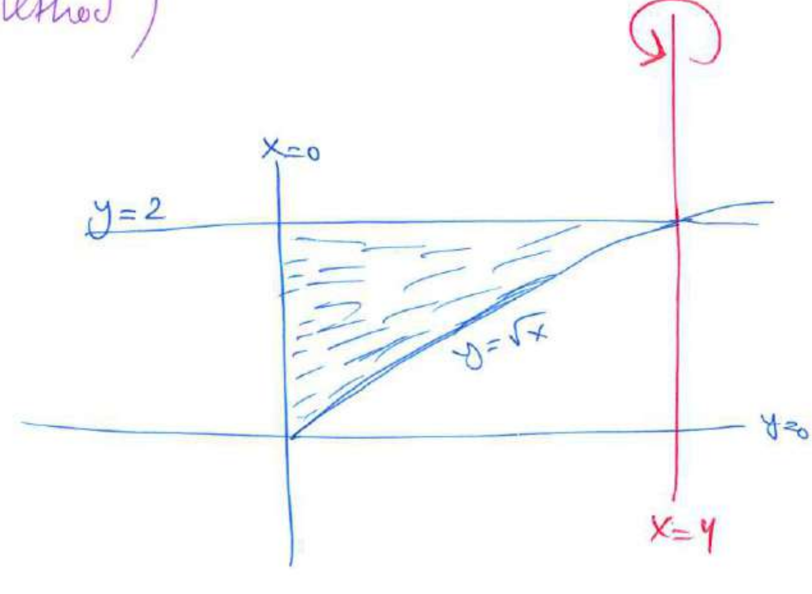
$R(x) = 2 - \sqrt{x}$

$R^2(x) = 4 - 4\sqrt{x} + x$

$$V = \int_0^4 \pi \left(4 - 4\sqrt{x} + x\right) dx$$

$$\pi \left[4x - \frac{8x^{3/2}}{3} + \frac{x^2}{2} \right]_0^4 = \boxed{\frac{8\pi}{3}}$$

(d) about $x=4$
(By using washer Method)



$$V = \int_{\square}^{\square} A(y)\, dy$$

$$V = \int_0^2 A(y)\, dy$$

$$= \int_0^2 \pi \left[R^2(y) - r^2(y)\right] dy$$

$R(y) = 4 - 0 = 4$

$r(y) = 4 - y^2$

$$= \int_0^2 \pi \left[16 - (4-y^2)^2\right] dy$$

$$= \pi \int_0^2 \cancel{16} - \cancel{16} + 8y^2 - y^4 \, dy = \pi \left[\frac{8y^3}{3} - \frac{y^5}{5}\right]_0^2$$

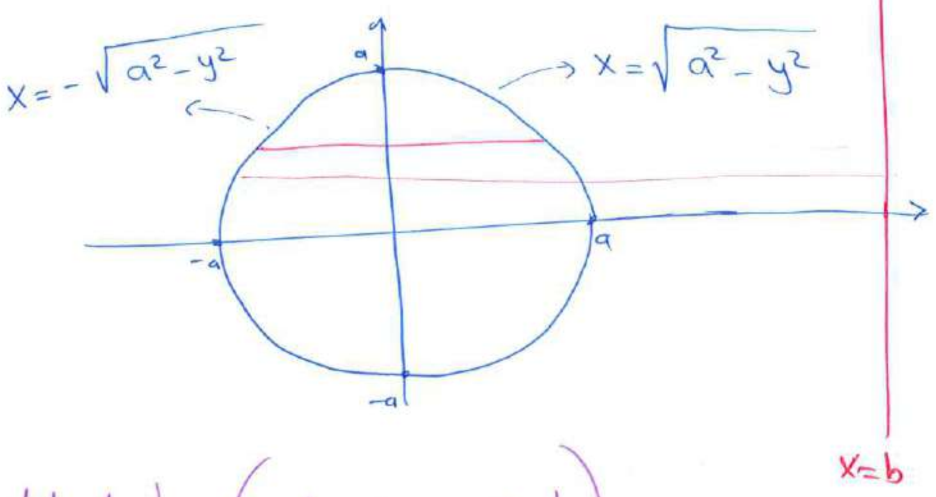
$$= \pi \left[\frac{64}{3} - \frac{32}{5}\right] = \frac{224\pi}{15}$$

Q51 The disk $x^2+y^2 \le a^2$ is revolved about the line $x=b$, $b>a$ to generate a solid shaped like a doughnut and called a torus. Find its volume

$$x^2+y^2=a^2$$

$$x^2=a^2-y^2$$

$$x=\pm\sqrt{a^2-y^2}$$



By using Washer Method (about $x=b$ vertical line)

$$V=\int_{\boxed{-a}}^{\boxed{a}} A(y)\,dy = \int_{-a}^{a} \pi\left[R^2(y)-r^2(y)\right]dy$$

$$=\pi\int_{-a}^{a}\left[b+\sqrt{a^2-y^2}\right]^2-\left[b-\sqrt{a^2-y^2}\right]^2 dy$$

$$=\pi\int_{-a}^{a}\left(\cancel{b^2}+2b\sqrt{a^2-y^2}+\cancel{a^2}-\cancel{y^2}-\cancel{b^2}+2b\sqrt{a^2-y^2}-\cancel{a^2}+\cancel{y^2}\right)dy$$

$$=\pi\int_{-a}^{a}4b\sqrt{a^2-y^2}\,dy$$

$$=4\pi b\int_{-a}^{a}\sqrt{a^2-y^2}\,dy$$

hint: $\int_{-a}^{a}\sqrt{a^2-y^2}\,dy=\frac{\pi a^2}{2}$

$$=4\pi b\cdot\frac{\pi a^2}{2}=\boxed{2\pi^2 ba^2}$$