

6.3 : Arc Length.

$$\text{Length} \quad L = \int_{\square}^{\square} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$L = \int_{\square}^{\square} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

2 Find the length of the curve $y = x^{3/2}$ from $x=0$ to $x=4$

$$\frac{dy}{dx} = \frac{3}{2} x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{9}{4} x$$

$$\left(\frac{dy}{dx}\right)^2 + 1 = \frac{9x + 4}{4}$$

$$L = \int_0^4 \sqrt{\frac{9x+4}{4}} dx$$

$$= \int_0^4 \frac{1}{2} \sqrt{9x+4} dx$$

$$= \int_4^{40} \frac{1}{2} \sqrt{u} \frac{du}{9} = \frac{1}{18} \int_4^{40} \sqrt{u} du$$

$$= \frac{1}{18} \cdot \frac{2}{3} \left[u^{3/2} \right]_4^{40}$$

$$= \frac{1}{27} \left[40^{3/2} - 4^{3/2} \right]$$

$$= \frac{1}{27} \left[8\sqrt{1000} - 8 \right] = \frac{8}{27} \left[\sqrt{1000} - 1 \right]$$

By substitution:-

$$\text{let } u = 9x + 4 \\ du = 9 dx$$

$$x=0 \rightarrow u=4$$

$$x=4 \rightarrow u=40$$

6 Find the length of the curve

$$x = \frac{y^3}{6} + \frac{1}{2y} \quad \text{from } y=2 \text{ to } y=3$$

$$\frac{dx}{dy} = \frac{3y^2}{6} + \frac{-1}{2y^2} = \frac{y^2}{2} - \frac{1}{2y^2}$$

$$\left(\frac{dx}{dy}\right)^2 = \frac{y^4}{4} - \frac{1}{2} + \frac{1}{4y^4}$$

$$\left(\frac{dx}{dy}\right)^2 + 1 = \frac{y^4}{4} + \frac{1}{2} + \frac{1}{4y^4}$$

$$L = \int_2^3 \sqrt{\frac{y^4}{4} + \frac{1}{2} + \frac{1}{4y^4}} dy$$

$$\int_2^3 \sqrt{\left(\frac{y^2}{2} + \frac{1}{2y^2}\right)^2} dy = \int_2^3 \left(\frac{y^2}{2} + \frac{1}{2y^2}\right) dy$$

$$= \left[\frac{y^3}{6} - \frac{1}{2y} \right]_2^3$$

$$\text{Length} = \frac{39}{12}$$

$$\boxed{7} \quad y = \frac{3}{4} x^{4/3} - \frac{3}{8} x^{2/3} + 5 \quad 1 \leq x \leq 8$$

$$L = \int_1^8 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$\frac{dy}{dx} = x^{1/3} - \frac{1}{4} x^{-1/3}$$

$$\left(\frac{dy}{dx}\right)^2 = x^{2/3} - \frac{2}{4} x^{1/3} x^{-1/3} + \frac{1}{16} x^{-2/3} = x^{2/3} - \frac{1}{2} + \frac{1}{16 x^{2/3}}$$

$$\left(\frac{dy}{dx}\right)^2 + 1 = x^{2/3} + \frac{1}{2} + \frac{1}{16 x^{2/3}}$$

$$L = \int_1^8 \sqrt{x^{2/3} + \frac{1}{2} + \frac{1}{16 x^{2/3}}} dx$$

$$L = \int_1^8 \sqrt{\left(x^{1/3} + \frac{1}{4 x^{1/3}}\right)^2} dx = \int_1^8 \left(x^{1/3} + \frac{1}{4 x^{1/3}}\right) dx$$

$$= \left[\frac{3}{4} x^{4/3} + \frac{3}{8} x^{2/3} \right]_1^8$$

$$= \frac{3}{4} \cdot 16 + \frac{3}{8} (4) - \frac{3}{4} - \frac{3}{8}$$

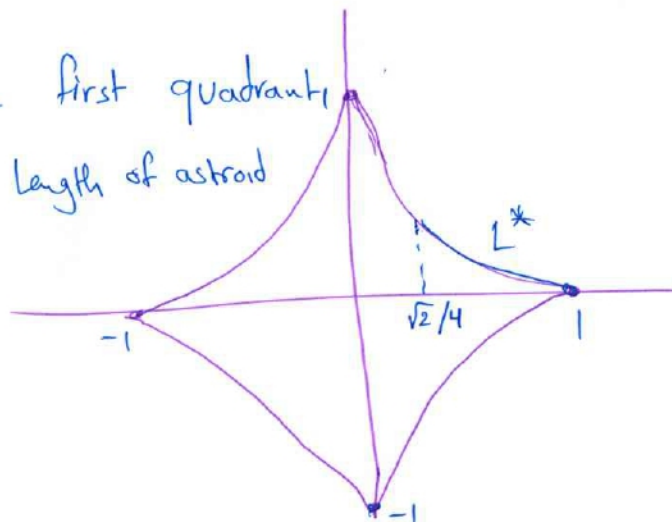
$$= 12 + \frac{3}{2} - \frac{3}{4} - \frac{3}{8}$$

$$= \frac{99}{8}$$

22 $x^{2/3} + y^{2/3} = 1$ is called astroid

• L^* : the length of half the first quadrant
 Portion \longrightarrow then to find the length of astroid

$$L = 8L^*$$



$$L^* = \int_{\frac{\sqrt{2}}{4}}^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$y = 1 - x^{2/3}$$

$$y = (1 - x^{2/3})^{3/2} \text{ first quadrant}$$

$$\frac{dy}{dx} = \frac{3}{2} (1 - x^{2/3})^{1/2} \cdot -\frac{2}{3} x^{-1/3}$$

$$= \frac{-\sqrt{1 - x^{2/3}}}{x^{1/3}}$$

solution

$$L = 8L^* = 8 \cdot \frac{3}{4} = 6$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1 - x^{2/3}}{x^{2/3}}$$

$$\left(\frac{dy}{dx}\right)^2 + 1 = \frac{1 - x^{2/3} + x^{2/3}}{x^{2/3}} = \frac{1}{x^{2/3}} = x^{-2/3}$$

$$L^* = \int_{\frac{\sqrt{2}}{4}}^1 \sqrt{x^{-2/3}} dx = \frac{3}{2} \cdot x^{2/3} \Big|_{\frac{\sqrt{2}}{4}}^1 = \frac{3}{2} \left[1 - \left(\frac{\sqrt{2}}{4}\right)^{2/3} \right] = \frac{3}{2} \left(1 - \frac{1}{2}\right) = \frac{3}{4}$$

6.4: Surface Area

15 Find the surface area that generated by revolving the curve $y = \sqrt{2x - x^2}$ about x -axis $\frac{1}{2} \leq x \leq \frac{3}{2}$

$$A = 2\pi \int_{\square}^{\square} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$\frac{dy}{dx} = \frac{2 - 2x}{2\sqrt{2x - x^2}} = \frac{1 - x}{\sqrt{2x - x^2}}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{(1-x)^2}{2x - x^2} = \frac{1 - 2x + x^2}{2x - x^2}$$

$$\left(\frac{dy}{dx}\right)^2 + 1 = \frac{1 - 2x + x^2 + 2x - x^2}{2x - x^2} = \frac{1}{2x - x^2}$$

$$A = 2\pi \int_{1/2}^{3/2} \sqrt{2x - x^2} \sqrt{\frac{1}{2x - x^2}} dx$$

$$A = 2\pi \int_{1/2}^{3/2} 1 dx$$

$$= 2\pi \left[x \right]_{1/2}^{3/2} = \boxed{2\pi}$$

17 Find the area of the surface generated by revolving the curve

$$x = \frac{y^3}{3}$$

$0 \leq y \leq 1$ about y -axis

$$A = 2\pi \int_{\square}^{\square} x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

$$\frac{dx}{dy} = y^2$$

$$\left(\frac{dx}{dy}\right)^2 = y^4$$

$$\left(\frac{dx}{dy}\right)^2 + 1 = y^4 + 1$$

$$A = 2\pi \int_0^1 \frac{y^3}{3} \sqrt{y^4 + 1} dy$$

$$\text{Let } u = y^4 + 1 \\ du = 4y^3 dy$$

$$y=0 \rightarrow u=1$$

$$y=1 \rightarrow u=2$$

$$A = 2\pi \int \frac{y^3}{3} \sqrt{u} \frac{du}{4y^3}$$

$$= \frac{2\pi}{12} \int_1^2 \sqrt{u} du$$

$$\frac{\pi}{6} \frac{2}{3} \left[u^{3/2} \right]_1^2$$

$$= \boxed{\frac{\pi}{9} [\sqrt{8} - 1]}$$

22 Find the area of the surface generated by revolving the curve $y = \frac{1}{3}(x^2+2)^{3/2}$ $0 \leq x \leq \sqrt{2}$ y-axis

Hint Express $ds = \sqrt{dx^2 + dy^2}$ in terms of dx , & evaluate the integral $S = \int 2\pi x ds$

$$y = \frac{1}{3}(x^2+2)^{3/2}$$

$$\frac{dy}{dx} = \frac{1}{2}(x^2+2)^{1/2} \cdot 2x = x(x^2+2)^{1/2}$$

$$dy = x(x^2+2)^{1/2} dx$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dx^2 + x^2(x^2+2)dx^2} = \sqrt{1+x^4+2x^2} dx$$

$$ds = \sqrt{1+x^4+2x^2} dx$$

$$S = \int 2\pi x ds$$

$$= \int_0^{\sqrt{2}} 2\pi x \sqrt{x^4+2x^2+1} dx$$

$$\int_0^{\sqrt{2}} 2\pi x \sqrt{(x^2+1)^2} dx$$

$$= 2\pi \int_0^{\sqrt{2}} x(x^2+1) dx$$
$$= 2\pi \left[\frac{x^4}{4} + \frac{x^2}{2} \right]_0^{\sqrt{2}}$$

$$= 2\pi [1+1-0] = \boxed{4\pi}$$