$\boxed{7.4}$ Exponential Change

$$\underset{\text{P.E}}{\text{I.V.P}} :- \quad \frac{dy}{dt} = ky \qquad \underset{\text{I.C}}{\quad} y = y_0 = y(t_0), \quad t_0 = 0$$

$$\frac{dy}{y} = k\,dt$$

$$\ln|y| = Kt + c$$

$$\longrightarrow \quad y = y_0 \, e^{kt} \qquad \text{Exponential growth}, \quad K > 0$$

$$\longrightarrow \quad y = y_0 \, e^{-kt} \qquad \text{Exponential decay}, \quad K > 0$$

$\boxed{Q_{28}}$ Voltage in a discharging capacitor. Suppose that electricity is draining from a capacitor at a rate that is proportional to the voltage across its terminals and that if t is measured in seconds $\dfrac{dV}{dt} = \dfrac{-1}{40} V$

Sol: Find V?? How long will it take the voltage to drop to 10% of the original value??

$$V = V_0 \, e^{-\frac{1}{40}t}$$

$$10\% . V_0 = V_0 \, e^{(-1/40)t}$$

$$0.10 = e^{(-1/40)t} \quad \longrightarrow \quad t = 40\ln 10 = \ln 10^{40} \ \text{sec}.$$

30. Growth of Bacteria

At the end of 3 hours there are 10000 bacteria

At the end of 5 hours there are 40000.

How many bacteria were present initially?

Sol:

$$y = y_0 \, e^{kt} \qquad \text{with} \qquad y(3) = 10000 \qquad y(5) = 40000$$

$$\rightarrow 10000 = y_0 \, e^{3k}$$
$$\rightarrow 40000 = y_0 \, e^{5k}$$

$$\Rightarrow 4 y_0 \, e^{3k} = y_0 \, e^{5k}$$

$$\Rightarrow e^{2k} = 4$$

$$\Rightarrow 2k = \ln 4 \Rightarrow \boxed{k = \ln 2}$$

$$y = y_0 \, e^{(\ln 2)t}$$

Now, Use $y(3) = 10000$ to find y_0

$$10000 = y_0 \, e^{3 \ln 2}$$

$$y_0 = \frac{10000}{8} = 1250 \text{ bacteria}$$

[31] The incidence of a disease :— (Continuation of Exp 3)

Suppose that in any given th number of cases can be reduced by 25% instead of 20%.

ⓐ How long will it take to reduce the number of cases to 1000 ??

We assume $y = y_0 e^{kt}$

If we count from today, then $y_0 = 10000$ at $t=0$

$$y = 10000 \, e^{kt}$$

$$7500 = 10000 \, e^{k}$$

$$0.75 = e^{k} \longrightarrow \boxed{k = \ln 0.75}$$

$K < 1$
exp. decay

Now, $y = y_0 e^{\ln 0.75 \, t}$ — — — Solve it for t.

$$1000 = 10000 \, e^{\ln 0.75 \, t}$$

$$\ln 0.1 = \ln 0.75 \; t$$

$$\boxed{t = 8 \text{ years}}$$

ⓑ How long will it take to eradicate the disease, that is, reduce the number of cases to less than 1 ??

$$y = 10000 \, e^{\ln 0.75 \, t}$$

$$1 = 10000 \, e^{\ln 0.75 \, t}$$

$$t = 32.02 \text{ years}$$

time: more than 32.02 years

36 The half Life of Polonium is 139 days but your Sample will not be useful to you after 95% of the Sample has disintegrated. For about How many days after the sample arrives will you be able to use the polonium ??

Sol: $y = y_0 e^{-kt}$

$I = $ half life $= 139$ day $\implies K = \dfrac{\ln 2}{I} = \dfrac{\ln 2}{139}$

$\implies K = 0.00499$

$y = y_0 e^{-0.00499 t}$

$0.05 \, y_0 = y_0 \, e^{-0.00499 t}$

$t = \dfrac{\ln 0.05}{-0.0049} = 600.34$

$$\boxed{t \approx 600 \text{ days}}$$

10 Use L'Hopital Rule to evaluate the limit :-

10 $\lim\limits_{t \to 1} \dfrac{t^3 - 1}{4t^3 - t - 3} = \dfrac{0}{0}$

$\lim\limits_{t \to 1} \dfrac{3t^2}{12t^2 - 1} = \dfrac{3}{11}$

21 $\lim\limits_{x \to 0} \dfrac{x^2}{\ln(\sec x)} = \dfrac{0}{0}$

$\lim\limits_{x \to 0} \dfrac{2x}{\dfrac{1}{\sec x} \sec x \tan x} = \lim\limits_{x \to 0} \dfrac{2x}{\tan x} = \lim\limits_{x \to 0} \dfrac{2}{\sec^2 x}$

$= \dfrac{2}{(1)^2} = 2$

27 $\lim\limits_{\theta \to 0} \dfrac{3^{\sin \theta} - 1}{\theta} = \dfrac{0}{0}$

$= \lim\limits_{\theta \to 0} \dfrac{3^{\sin \theta} (\ln 3)(\cos \theta)}{1}$

$= (\ln 3)(3^0)(1) = \ln 3$

30

$$\lim_{x \to 0} \frac{3^x - 1}{2^x - 1} = \frac{0}{0}$$

$$\lim_{x \to 0} \frac{3^x \ln 3}{2^x \ln 2} = \frac{\ln 3}{\ln 2} = \boxed{\log_2 3}$$

34

$$\lim_{x \to 0^+} \frac{\ln(e^x - 1)}{\ln x} = \frac{-\infty}{-\infty}$$

$$\lim_{x \to 0^+} \frac{e^x}{e^x - 1} \Big/ \frac{1}{x}$$

$$= \lim_{x \to 0^+} \frac{x e^x}{e^x - 1}$$

$$= \lim_{x \to 0^+} \frac{x e^x + e^x}{e^x} = \lim_{x \to 0^+} x + 1 = \boxed{1}$$

38

$$\lim_{x \to 0^+} \left(\ln x - \ln \sin x \right) = -\infty + \infty = \infty - \infty$$

Another solution:-

$$= \lim_{x \to 0^+} \ln \left(\frac{x}{\sin x} \right)$$

$$\lim_{x \to 0^+} \frac{x/x}{\sin x / x}$$

$$= \ln \left[\lim_{x \to 0^+} \frac{x}{\sin x} \right] = \frac{0}{0}$$

$$= \lim_{x \to 0^+} \frac{1}{\frac{\sin x}{x}}$$

$$= \ln \left[\lim_{x \to 0^+} \frac{1}{\cos x} \right] = \ln(1) = \boxed{0}$$

$$= \frac{1}{1} = \boxed{1}$$

$$\boxed{\ln(1) = 0}$$

41 $\lim\limits_{x \to 1^+} \left(\dfrac{1}{x-1} - \dfrac{1}{\ln x} \right)$ $\dfrac{1}{small\,+} - \dfrac{1}{small\,4}$

$\infty - \infty$

$= \lim\limits_{x \to 1^+} \dfrac{\ln x - x + 1}{\ln x (x-1)}$ $\dfrac{0}{0}$

$= \lim\limits_{x \to 1^+} \dfrac{\frac{1}{x} - 1}{\ln x (1) + (x-1) \cdot \frac{1}{x}}$

$= \lim\limits_{x \to 1^+} \dfrac{1-x}{x \ln x + x - 1}$ $= \dfrac{0}{0}$

$= \lim\limits_{x \to 1^+} \dfrac{-1}{\frac{x}{x} + \ln x + 1}$

$= \lim\limits_{x \to 1^+} \dfrac{-1}{2 + \ln x}$ $= \dfrac{-1}{2 + 0}$ $= \boxed{\dfrac{-1}{2}}$

42 $\lim\limits_{x \to 0^+} \left(\csc x - \cot x + \cos x \right)$ $\dfrac{1}{small\,+} - \dfrac{1}{small\,+} + 1$

$\infty - \infty + 1$

$= \lim\limits_{x \to 0^+} \left(\dfrac{1}{\sin x} - \dfrac{\cos x}{\sin x} \right) + \lim\limits_{x \to 0^+} \cos x$

$= \lim\limits_{x \to 0^+} \dfrac{1 - \cos x}{\sin x} \; \frac{0}{0} \quad + \lim\limits_{x \to 0^+} \cos x$

$= \lim\limits_{x \to 0^+} \dfrac{+\sin x}{\cos x} \quad + \lim\limits_{x \to 0^+} \cos x$

$= \quad 0 \quad + \quad 1 \quad = \boxed{1}$

46) $\lim\limits_{x \to \infty} x^2 e^{-x}$ $\infty \cdot 0$

$$= \lim\limits_{x \to \infty} \frac{x^2}{e^x} \quad \frac{\infty}{\infty}$$

$$= \lim\limits_{x \to \infty} \frac{2x}{e^x}$$

$$= \lim\limits_{x \to \infty} \frac{2}{e^x} = \frac{2}{\infty} = \boxed{0}$$

47) $\lim\limits_{x \to 0} \dfrac{x - \sin x}{x \tan x}$ $\dfrac{0}{0}$

$$= \lim\limits_{x \to 0} \frac{1 - \cos x}{x \sec^2 x + \tan x} = \frac{0}{0}$$

$$= \lim\limits_{x \to 0} \frac{\sin x}{2x \cdot \sec^2 x \tan x + \sec^2 x + \sec^2 x}$$

$$= \frac{0}{0+2} = \boxed{0}$$

52 $\lim\limits_{x\to 1^+} x^{1/x-1}$ $(1)^\infty$

* $f(x) = x^{\frac{1}{x-1}}$

$\longrightarrow \ln f(x) = \dfrac{1}{x-1}\ln x$

$\longrightarrow \lim\limits_{x\to 1^+} \ln f(x) = \lim\limits_{x\to 1^+} \dfrac{\ln x}{x-1} \quad \dfrac{0}{0}$

$= \lim\limits_{x\to 1^+} \dfrac{\frac{1}{x}}{1}$

$= 1$

$\therefore \lim\limits_{x\to 1^+} x^{\frac{1}{x-1}} = e^1 = \boxed{e}$

53 $\lim\limits_{x\to\infty} (\ln x)^{1/x}$

* $f(x) = (\ln x)^{\frac{1}{x}}$

\# $\ln f(x) = \dfrac{1}{x}\ln(\ln x)$

* $\lim\limits_{x\to\infty} \ln f(x) = \lim\limits_{x\to\infty} \dfrac{\ln(\ln x)}{x} \quad \dfrac{\infty}{\infty}$

$= \lim\limits_{x\to\infty} \dfrac{\frac{1}{\ln x}\cdot\frac{1}{x}}{1} = 0$

$\therefore \lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} (\ln x)^{1/x} = e^0 = 1$

$$\boxed{60} \quad \lim_{x \to 0^+} \left(1 + \frac{1}{x}\right)^x = \left(\infty^0\right)$$

$$* \quad f(x) = \left(1 + \frac{1}{x}\right)^x$$

$$* \quad \ln f(x) = x \, \ln\left(1 + \frac{1}{x}\right)$$

$$* \lim_{x \to 0^+} \ln f(x) = \lim_{x \to 0^+} \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}} = \frac{\infty}{\infty}$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{1 + 1/x} \cdot \frac{-1}{x^2}}{\frac{-1}{x^2}}$$

$$= \lim_{x \to 0^+} \frac{x}{x+1}$$

$$= 0$$

$$\therefore \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(1 + \frac{1}{x}\right)^x = e^0 = 1$$

Theorem $\quad \lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = e^a$

$\boxed{62}$ $\lim\limits_{x \to \infty} \left(\dfrac{x^2 + 1}{x + 2} \right)^{1/x}$ $\quad (\infty^0)$

* $f(x) = \left(\dfrac{x^2 + 1}{x + 2} \right)^{1/x}$

* $\ln f(x) = \dfrac{1}{x} \ln \left(\dfrac{x^2 + 1}{x + 2} \right)$

* $\lim\limits_{x \to \infty} (\ln f(x)) = \lim\limits_{x \to \infty} \dfrac{\ln \left(\dfrac{x^2 + 1}{x + 2} \right)}{x}$

$\qquad = \lim\limits_{x \to \infty} \dfrac{\ln(x^2 + 1) - \ln(x + 2)}{x} \qquad \dfrac{\infty}{\infty}$

$\qquad = \lim\limits_{x \to \infty} \dfrac{\dfrac{2x}{x^2 + 1} - \dfrac{1}{x + 2}}{1}$

$\qquad = \lim\limits_{x \to \infty} \dfrac{2x(x+2) - (x^2 + 1)}{(x^2 + 1)(x + 2)} \qquad \dfrac{\infty}{\infty}$

$\qquad = \lim\limits_{x \to \infty} \dfrac{2x^2 + 4x - x^2 - 1}{x^3 + 2x^2 + x + 2}$

$\qquad = \lim\limits_{x \to \infty} \dfrac{4x + 4 - 2x}{3x^2 + 4x + 1} = \lim\limits_{x \to \infty} \dfrac{2x + 4}{3x^2 + 4x + 1}$

$\qquad = \lim\limits_{x \to \infty} \dfrac{2}{6x + 4} = 0 \qquad \therefore \lim\limits_{x \to \infty} f(x) = e^0 = \boxed{1}$

$\boxed{66}$ $\displaystyle\lim_{x \to 0^+}$ $(\sin x)(\ln x)$ $\qquad 0 \cdot \bar{\infty}$

$= \displaystyle\lim_{x \to 0^+} \dfrac{\ln x}{\csc x} \qquad \dfrac{-\infty}{\infty}$

$= \displaystyle\lim_{x \to 0^+} \dfrac{\dfrac{1}{x}}{-\csc x \cot x}$

$= \displaystyle\lim_{x \to 0^+} \dfrac{1/x}{\dfrac{-1}{\sin x} \cdot \dfrac{1}{\tan x}}$

$= \displaystyle\lim_{x \to 0^+} \dfrac{-\sin x \, \tan x}{x} \qquad \dfrac{0}{0}$

$\boxed{\text{Th.} \quad \displaystyle\lim_{x \to 0} \dfrac{\sin x}{x} = 1}$

$= -\displaystyle\lim_{x \to 0^+} \dfrac{\sin x}{x} \cdot \displaystyle\lim_{x \to 0^+} \tan x$

$= \qquad -1 \qquad \cdot \quad 0 \quad = \boxed{0}$

72

$$\lim_{x \to -\infty} \frac{2^x + 4^x}{5^x - 2^x} \qquad \frac{0}{0} \quad \text{l'Hôpital Rule}$$

$$= \lim_{x \to -\infty} \frac{2^x \left[1 + \left(\frac{4}{2} \right)^x \right]}{2^x \left[\left(\frac{5}{2} \right)^x - 1 \right]}$$

$$= \lim_{x \to -\infty} \frac{1 + \left(\frac{4}{2} \right)^x}{\left(\frac{5}{2} \right)^x - 1}$$

$$= \lim_{x \to -\infty} \frac{1 + 2^x}{\left(\frac{5}{2} \right)^x - 1} = \frac{1 + 0}{0 - 1} = \boxed{-1}$$

Another solution:-

$$\lim_{x \to -\infty} \frac{2^x + 4^x}{5^x - 2^x} = \lim_{x \to -\infty} \frac{2^x \ln 2 + 4^x \ln 4}{5^x \ln 5 \div 2^x \ln 2}$$

$$= \lim_{x \to -\infty} \frac{\ln 2 + 2^x \ln 4}{\left(\frac{5}{2} \right)^x \ln 5 - \ln 2}$$

$$= \frac{\ln 2}{-\ln 2} = -1$$