

7.8 Relative Rate of Growth

[2] Which of the following functions grow faster than e^x as $x \rightarrow \infty$ which grow at the same rate as e^x which grow slower

(a) $10x^4 + 30x + 1$

$$\lim_{x \rightarrow \infty} \frac{e^x}{10x^4 + 30x + 1} = \frac{\infty}{\infty}$$

$$= \lim_{x \rightarrow \infty} \frac{e^x}{40x^3 + 30}$$

$$= \lim_{x \rightarrow \infty} \frac{e^x}{120x^2}$$

$$= \lim_{x \rightarrow \infty} \frac{e^x}{240x}$$

$$= \lim_{x \rightarrow \infty} \frac{e^x}{240} = \infty \quad e^x \text{ grows faster than } 10x^4 + 30x + 1 \text{ as } x \rightarrow \infty$$

$$(b) \quad x \ln x - x$$

$$\lim_{x \rightarrow \infty} \frac{e^x}{x \ln x - x} =$$

$$\lim_{x \rightarrow \infty} \frac{e^x}{x(\ln x - 1)} = \frac{\infty}{\infty}$$

$$\lim_{x \rightarrow \infty} \frac{e^x}{x \left(\frac{1}{x} + (\ln x - 1) \right)} = \lim_{x \rightarrow \infty} \frac{e^x}{\ln x}$$

$$= \lim_{x \rightarrow \infty} \frac{e^x}{\frac{1}{x}}$$

$$= \lim_{x \rightarrow \infty} x e^x = \infty$$

e^x grows faster than $x \ln x - x$

$$(c) \quad \sqrt{1+x^4}$$

$$\lim_{x \rightarrow \infty} \frac{\sqrt{1+x^4}}{e^x} = \lim_{x \rightarrow \infty} \sqrt{\frac{1+x^4}{e^{2x}}} = \sqrt{\lim_{x \rightarrow \infty} \frac{1+x^4}{e^{2x}}} \quad \frac{\infty}{\infty}$$

$$= \sqrt{\lim_{x \rightarrow \infty} \frac{4x^3}{2e^{2x}}}$$

$$= \sqrt{\lim_{x \rightarrow \infty} \frac{12x^2}{4e^{2x}}} = \sqrt{\lim_{x \rightarrow \infty} \frac{24x}{8e^{2x}}}$$

$$\lim_{x \rightarrow \infty} \sqrt{\frac{1+x^4}{e^{2x}}} = \sqrt{\lim_{x \rightarrow \infty} \frac{24}{16 e^{2x}}} = 0$$

$\sqrt{1+x^4}$ grows slower than e^x as $x \rightarrow \infty$

d) $\left(\frac{5}{2}\right)^x$

$$\lim_{x \rightarrow \infty} \frac{\left(\frac{5}{2}\right)^x}{e^x} = \lim_{x \rightarrow \infty} \left(\frac{2.5}{e}\right)^x = 0$$

$\left(\frac{5}{2}\right)^x$ grows slower than e^x as $x \rightarrow \infty$

e) e^{-x}

$$\lim_{x \rightarrow \infty} \frac{e^{-x}}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{e^{2x}} = 0$$

e^{-x} grows slower than e^x as $x \rightarrow \infty$

f) $x e^x$

$$\lim_{x \rightarrow \infty} \frac{x e^x}{e^x} = \infty$$

$x e^x$ grows faster than e^x as $x \rightarrow \infty$

$$\textcircled{g} \quad e^{\cos x}$$

$$\lim_{x \rightarrow \infty} \frac{e^{\cos x}}{e^x} = \lim 0 = 0$$

$$-1 \leq \cos x \leq 1$$

$$e^{-1} \leq e^{\cos x} \leq e^1$$

By Sandwich Th.

$$\lim_{x \rightarrow \infty} \frac{e^{-1}}{e^x} \leq \lim_{x \rightarrow \infty} \frac{e^{\cos x}}{e^x} \leq \lim_{x \rightarrow \infty} \frac{e^1}{e^x}$$

\downarrow
0

$$\lim_{x \rightarrow \infty} \frac{e^{\cos x}}{e^x} = 0$$

Then e^x grows faster than $e^{\cos x}$

$$\textcircled{h} \quad e^{x-1}$$

$$\lim_{x \rightarrow \infty} \frac{e^{x-1}}{e^x} = \lim_{x \rightarrow \infty} e^{-1} = \frac{1}{e} = \text{constant}$$

Then e^{x-1} and e^x grows at the same rate as $x \rightarrow \infty$

Q5 Which of the following functions grow faster than $\ln x$ as $x \rightarrow \infty$? which grow slower?

(a) $\log_3 x$

$$\lim_{x \rightarrow \infty} \frac{\log_3 x}{\ln x} = \lim_{x \rightarrow \infty} \frac{\ln x / \ln 3}{\ln x} = \frac{1}{\ln 3}$$

$\log_3 x \approx \ln x$ grows at the same rate as $x \rightarrow \infty$

(b) $\ln 2x$

$$\lim_{x \rightarrow \infty} \frac{\ln 2x}{\ln x}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{2x} \cdot (2)}{\frac{1}{x}} = 1$$

$\ln 2x \approx \ln x$ grows at the same rate as $x \rightarrow \infty$

(c) $\ln \sqrt{x}$

$$\lim_{x \rightarrow \infty} \frac{\ln \sqrt{x}}{\ln x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}}{\frac{1}{x}}$$

$$= \frac{1}{2}$$

$\ln \sqrt{x} \approx \ln x$ grows at the same rate as $x \rightarrow \infty$

d) \sqrt{x}

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\ln x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{2} = \infty$$

\sqrt{x} grows faster than $\ln x$ as $x \rightarrow \infty$

e) x

$$\lim_{x \rightarrow \infty} \frac{x}{\ln x} = \lim_{x \rightarrow \infty} \frac{1}{\frac{1}{x}} = \infty$$

x grows faster than $\ln x$ as $x \rightarrow \infty$

f) $5 \ln x$

$$\lim_{x \rightarrow \infty} \frac{5 \ln x}{\ln x} = 5$$

$5 \ln x$ & $\ln x$ grows at the same rate as $x \rightarrow \infty$

g) $\frac{1}{x}$

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\ln x} = \lim_{x \rightarrow \infty} \frac{1}{x \ln x} = 0$$

$\frac{1}{x}$ grows slower than $\ln x$ as $x \rightarrow \infty$

(h) e^x

$$\lim_{x \rightarrow \infty} \frac{e^x}{\ln x} = \lim_{x \rightarrow \infty} \frac{e^x}{\frac{1}{x}} = \infty$$

e^x grows faster than $\ln x$ as $x \rightarrow \infty$

Q8 Order the following function from slowest growing to fastest growing as $x \rightarrow \infty$

Solution $(\ln 2)^x, x^2, 2^x, e^x$ since.

$$\lim_{x \rightarrow \infty} \frac{e^x}{2^x} = \lim_{x \rightarrow \infty} \left(\frac{e}{2}\right)^x = \infty$$

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{2^x}{x^2} &= \lim_{x \rightarrow \infty} \frac{2^x \ln 2}{2x} \\ &= \lim_{x \rightarrow \infty} \frac{2^x (\ln 2)^2}{2} \\ &= \infty \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{(\ln 2)^x}{x^2} &= \lim_{x \rightarrow \infty} \frac{1}{x^2} \lim_{x \rightarrow \infty} (\ln 2)^x, \\ &= 0 \cdot 0 \\ &= 0 \end{aligned}$$

