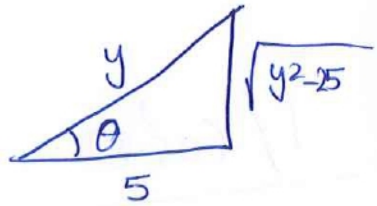
8.3 Trigonometric Substitutions

12 $$\int \frac{\sqrt{y^2-25}}{y^3}\, dy$$



let $y = 5\sec\theta$

$dy = 5\sec\theta\tan\theta$

$$\int \frac{\sqrt{25\sec^2\theta - 25}}{125\sec^3\theta}\, 5\sec\theta\tan\theta\, d\theta$$

$$\int \frac{5\tan\theta \cdot 5\sec\theta\tan\theta}{125\sec^3\theta}\, d\theta$$

$$\int \frac{25\tan^2\theta}{125\sec^2\theta}\, d\theta = \int \frac{25\sin^2\theta/\cos^2\theta}{125/\cos^2\theta}\, d\theta$$

$$= \int \frac{\sin^2\theta}{5}\, d\theta$$

$$= 1/5 \int \frac{1-\cos(2\theta)}{2}\, d\theta$$

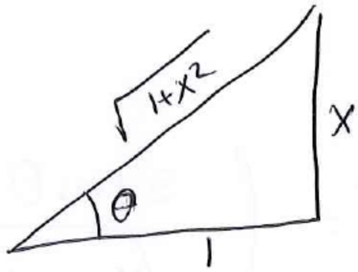
$$= \frac{1}{10}\left[\theta - \frac{\sin(2\theta)}{2} + C\right]$$

$$I = \frac{1}{10}\left[\sec^{-1}\left(\frac{y}{5}\right) - \frac{\sqrt{y^2-25}}{y} \cdot \frac{5}{y} + C\right.$$

18 $$\int \frac{dx}{x^2\sqrt{x^2+1}}$$

let $x = \tan\theta$

$dx = \sec^2\theta \, d\theta$



$$= \int \frac{\sec^2\theta \, d\theta}{\tan^2\theta \sqrt{\tan^2\theta + 1}}$$

$$\int \frac{d\theta}{\sin^2\theta \sqrt{\sec^2\theta}}$$

$$= \int \frac{d\theta}{\sin^2\theta \sec\theta} = \int \frac{\cos\theta}{\sin^2\theta} \, d\theta$$

let $u = \sin\theta$

$du = \cos\theta \, d\theta$

$$= \int \frac{1}{u^2} \, du$$

$$= \frac{-1}{u} + C$$

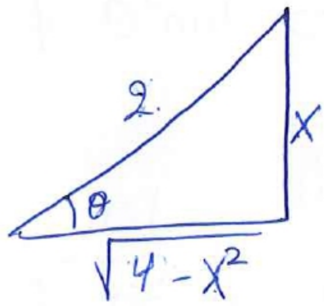
$$= \frac{-1}{\frac{x}{\sqrt{1+x^2}}} + C$$

$$= \boxed{\frac{-\sqrt{1+x^2}}{x} + C}$$

$\boxed{24}$ $\displaystyle\int_0^1 \frac{dx}{(4-x^2)^{3/2}}$

let $x = 2\sin\theta$

$dx = 2\cos\theta \, d\theta$



$$I = \int \frac{2\cos\theta \, d\theta}{(4 - 4\sin^2\theta)^{3/2}}$$

$$= \int \frac{2\cos\theta \, d\theta}{(4\cos^2\theta)^{3/2}}$$

$$= \int \frac{2\cos\theta \, d\theta}{8\cos^3\theta}$$

$$= \frac{1}{4}\int \frac{1}{\cos^2\theta} d\theta = \frac{1}{4}\int \sec^2\theta \, d\theta$$

$$= \frac{1}{4}\tan\theta$$ من المثلث

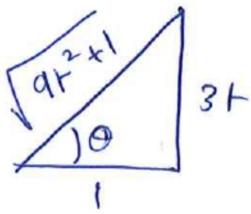
$$= \frac{1}{4} \left. \frac{x}{\sqrt{4-x^2}} \right]_0^1$$

$$= \frac{1}{4}\left[\frac{1}{\sqrt{3}} - 0\right] = \frac{1}{4\sqrt{3}}$$

$\boxed{30}$ $$\int \frac{6}{(9t^2+1)^2}\, dt$$

let $3t = \tan\theta$

$3\,dt = \sec^2\theta \, d\theta$



$$I = \int \frac{6}{(\tan^2\theta + 1)^2} \frac{\sec^2\theta}{3}\, d\theta$$

$$= 2\int \frac{\sec^2\theta}{(\sec^2\theta)^2}\, d\theta$$

$$= 2\int \frac{1}{\sec^2\theta}\, d\theta$$

$$= 2\int \cos^2\theta \, d\theta$$

$$= \cancel{2}\int \frac{1+\cos(2\theta)}{\cancel{2}}\, d\theta$$

$$= \theta + \frac{\sin(2\theta)}{2} + C$$

$$= \tan^{-1}(3t) + \frac{1}{2}\cdot 2\sin\theta\cos\theta + C$$

$$= \tan^{-1}(3t) + \frac{3t}{\sqrt{9t^2+1}} \cdot \frac{1}{\sqrt{9t^2+1}} + C$$

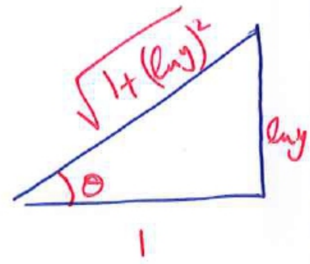
$$= \tan^{-1}(3t) + \frac{3t}{9t^2+1} + C$$

$\boxed{38}$ $$\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}}$$

let $\ln y = \tan\theta$

$$\frac{1}{y}\,dy = \sec^2\theta \, d\theta$$

$$dy = y \sec^2\theta \, d\theta$$



$$\int \frac{y \sec^2\theta \, d\theta}{y\sqrt{1+(\tan\theta)^2}}$$

$$I = \int \frac{\sec^2\theta \, d\theta}{\sqrt{1+\tan^2\theta}}$$

$$= \int \frac{\sec^2\theta \, d\theta}{\sec\theta} = \int \sec\theta \, d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| \Big]_{\square}^{\square}$$

$$= \ln\left|\sqrt{1+\ln^2 y} + \ln y\right| \Big]_1^e$$

$$= \ln\left|\sqrt{2}+1\right| - \ln|1| = \ln\left(\sqrt{2}+1\right).$$