lec 40

7.4 Exponensial change and separable differential equations

- Differential equations (DE) are equations with derivatives (changes, rates) → relations

→ initial condition (IC)

↳ $y(t_0) = y_0$

→ initial value problem IVP: is a DE with IC

Remark: We may use the seprable Method to solve some DE's

Exp: solve the DE: $y' = \frac{dy}{dx} = x^2\sqrt{y}$

$$\Rightarrow \int \frac{2\,dy}{2\sqrt{y}} = \int x^2\,dx$$

$$\Rightarrow 2\sqrt{y} = \frac{x^3}{3} + C$$

$$\to \sqrt{y} = \frac{2x^3}{6} + \frac{c}{2}$$

$$y(x) = \left(\frac{x^3}{6} + \frac{c}{2}\right)^2$$

this is an Explicit solution

Exp:- solve the initial value Problem

$$\frac{dy}{dt} = \frac{y \cos(t)}{1+3y^3}, \quad y(0) = 1$$

$$\int \frac{(1+3y^3)}{y}\, dy = \int \cos(t)\, dt$$

$$\Rightarrow \int \frac{1}{y}\, dy + \int 3y^2\, dy = \sin(t) + c$$

$$\ln|y| + \frac{\cancel{3}y^3}{\cancel{3}} = \sin(t) + c$$

$$\ln|y| + y^3 = \sin(t) + c$$

this is an implicit solution

To find C:-

$$\ln(1) + 1^3 = \sin 0 + C$$

$$0 + 1 = 0 + C$$

$$\boxed{C = 1}$$

$$\Rightarrow \ln|y| + y^3 = \sin(t) + 1$$

Remark:- we usually use DE's to model several phenomena:

Exp:- (Growth Rate)

- Assume a population P increases exponensially to it's current size P. Write a DE that describes the population size over time.

So Assume $P(0) = P_0$

$$\boxed{\frac{dP}{dt} \propto P(t) = kP, \quad P(0) = 0}$$

IVP ①

$\propto$: propotion

k : constant

$k > 0$ → Growth rate

• How to solve the IVP ①?

→ We use the method of calculus (separation) :-

$$\cancel{\int \frac{dP}{dt} = \int k\, dt}$$

$$\int \frac{dP}{P} = \int k\, dt$$

$$\ln|P| = kt + C$$

$$|P| = e^{kt+C}$$

$$|P(t)| = e^{C} e^{kt}$$ constant D

$$P(t) = \pm e^{C} e^{kt}$$

$$P(t) = D e^{kt}$$

How to find D :? we use IC

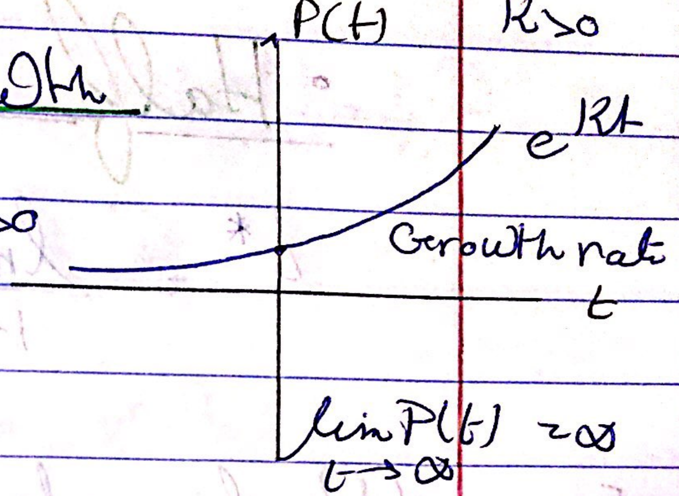
$$P(0) = D e^{k0} = P_0$$

$$D = P_0$$

Hence, solution is

$$\boxed{P(t) = P_0 e^{kt}}$$

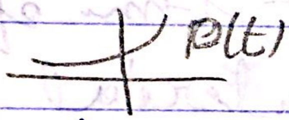
• Population Growth



DE:- $\dfrac{dP}{dt} = kP$ $\quad k>0$

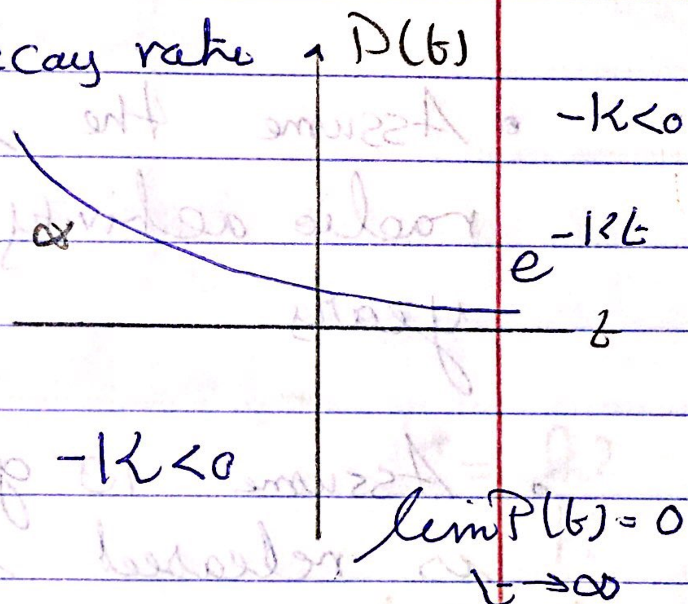
IC:- $P(t_0) = P_0$

Solution:- $P(t) = P_0 e^{Kt}$

Graph 

$\lim_{t \to \infty} P(t) = \infty$

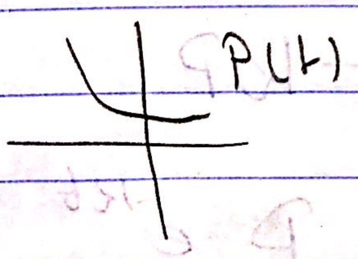
• Population decay



DE: $\dfrac{dP}{dt} = -KP$ $\quad -K<0$

IC:- $P(t_0) = P_0$

solution $P(t) = P_0 e^{-Kt}$

Graph 

$\lim_{x \to \infty} P(t) = 0$

• Half life time

$$t^* = \frac{\ln 2}{k}$$

Exp (Radioactivity) :-

- Assume the half-life time of a radio activity material is ln5 years
- Assume 10 gm of this material is released to atmosphere
- How many years we this material needs to 80% to decay

$$\frac{dP}{dt} = -kP$$

$$P(t) = P_0 e^{-kt}$$

find t^* s.t : $P(t^*) = \frac{1}{2} P_0$

$$\Rightarrow \cancel{P_0}(e^{-kt^*}) = \frac{1}{2}\cancel{P_0}$$

$$\cancel{-}(k)t^* = \cancel{-}\ln 2$$

$$(k)(t^*) = \ln(2)$$

$$t^* = \frac{\ln 2}{k}$$

• This Example is a decay :-

$$P(t) = P_0 e^{-kt}$$

$$t^* = \frac{\ln 2}{K}$$

$$P(t) = P_0 e^{-\frac{1}{3}t}$$

$$\ln 8 = \frac{\ln 2}{k}$$

But $P_0 = 10$ gm

$$k = \frac{\ln 2}{3\ln 2}$$

$$P(t) = 10 e^{(-\frac{1}{3})t}$$

$$k = \frac{1}{3}$$

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find $\bar{t}$ s.t $P(\bar{t}) = \frac{20}{100} \times 10$

$= 2$ gm

$$10 e^{(-\frac{1}{3})\bar{t}} = 2$$

$$e^{(-\frac{1}{3})\bar{t}} = 0.2$$

$$-\frac{1}{3}\bar{t} = \ln 0.2 \Rightarrow -\frac{1}{3}\bar{t} = -\ln 5$$

$$\bar{t} = 3\ln 5 \text{ year}$$

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