CH2 : Limits and Continuity.

2.1 : Limits of functions :

$\lim\limits_{x \to x_0} (f(x)) = L$ "Limit f(x) as X goes to X_0 is equal to L".

limit $f(x) = L$ if and only if
$X \longrightarrow X_0$

$$\lim_{x \to x_0^{+}} f(x) = \lim_{x \to x_0^{-}} f(x) = L$$

ex : find all the limits:

ⓐ: $\lim\limits_{x \to 1} \dfrac{x-1}{x+1} = \dfrac{1-1}{1+1} = \dfrac{0}{2} = 0$ (exists).

ⓑ: $\lim\limits_{x \to 1} \dfrac{x^2-1}{x-1} = \left(\dfrac{0}{0}\right)$

$\lim\limits_{x \to 1} \dfrac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} \quad \lim\limits_{x \to} x+1 = 2$ (exists).

ⓒ: $\lim\limits_{x \to \infty} \dfrac{1}{x} = 0$

$\lim\limits_{x \to -\infty} \dfrac{1}{x} = 0$

exists.

d: $\lim_{x \to 0^+} \frac{1}{x} = \infty$

$\lim_{x \to 0^-} \frac{1}{x} = -\infty$ doesn't exist. $\infty \neq -\infty$.

(DNE)

غير موجودة * $\lim_{x \to \infty^+}$ نها $= \infty$ غير موجودة لأن ∞ عدد غير حقيقي.

e: $\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \frac{0}{0}$

$\lim_{x \to 1} \frac{(x+2)\cancel{(x-1)}}{x\cancel{(x-1)}} = \frac{x+2}{x} = \frac{3}{1} = 3$ ∴ exists.

f: $\lim_{x \to 0} \frac{\sqrt{x^2+8} - 3}{x+1} = \sqrt{8} - 3$

* $\lim_{x \to -1} \frac{\sqrt{x^2+8} - 3}{x+1} = \frac{0}{0}$

نضرب بالمرافق

Conjugate $\lim_{x \to -1} \frac{\sqrt{x^2+8} - 3}{x+1} \times \frac{\sqrt{x^2+8} + 3}{\sqrt{x^2+8} + 3}$

$\lim_{x \to -1} = \frac{x^2 + 8 + 9}{(x+1)(\sqrt{x^2+8} + 3)} = \lim_{x \to -1} \frac{(x-1)(x+1)}{(x+1)(\sqrt{x^2+8} + 3)}$

$= \frac{-2}{6} = \frac{-1}{3}$.

The Sandwich Theorem :

if $f(x) \leq g(x) \leq h(x)$

and $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$

then $\lim_{x \to c} g(x) = L$

ex : if $1 - x^2 \leq f(x) \leq 1 + x^2$

find $\lim_{x \to 0} f(x)$

Sol: $\lim_{x \to 0} (1 - x^2) = 1$

$\lim_{x \to 0} (1 + x^2) = 1$

$\therefore \lim_{x \to 0} f(x) = 1$, by Sandwich Thm.

$$-1 \leq \sin x \leq 1$$

$$\frac{-1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

$$\lim_{x \to \infty} \frac{-1}{x} = 0 = \lim_{x \to \infty} \frac{1}{x}$$

$$\Rightarrow \lim_{x \to \infty} \frac{\sin x}{x} = 0$$

$$* \lim_{x \to -\infty} \frac{\sin x}{x} = 0$$

ex: $\lim\limits_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

$\lim\limits_{x \to 0} x^2 = 0$ and $\lim\limits_{x \to 0} (-x^2) = 0$

$\therefore \lim\limits_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$ by the squeeze Thm.

ex: if $x^3 - \cancel{x} + 4 \leq f(x) + x \leq 3x^2 + 1$
for all real numbers x
find $\lim\limits_{x \to 0} f(x)$

حتى نطبق على
نظرية ساندوتش
يجب طرح
X

Solution: $x^3 - 2x + 4 \leq f(x) \leq 3x^2 - x + 1$

Now: $\lim\limits_{x \to 1} (x^3 - 2x + 4) = 1 - 2 + 4 = 3$

$\lim\limits_{x \to 1} (3x^2 - x + 1) = 3 - 1 + 1 = 3$

$\therefore \lim\limits_{x \to 1} f(x) = 3$ by Sandwich Thm.

[2.2] Continuity الاتصال

f is continous at $x = x_0$ if :

①: $f(x_0)$ exists (defined).

②: $\lim\limits_{x \to x_0} f(x)$ exists.

③: $f(x_0) = \lim\limits_{x \to x_0} f(x)$

ex: find if f is continous?

①: $f(x) = \sin x$ or $\cos x$ are continous on $(-\infty, \infty)$.

②: $y = |x|$, polynomials are continous on $(-\infty, \infty)$.

③: $y = \sec x$ or $y = \tan x$ are continous on $(-\infty, \infty) \setminus \{\frac{\pi}{2} + n\pi, n = 0, \pm 1, \pm 2 \ldots\}$ ماعدا

ex: $f(x) = \dfrac{x^2 - 4}{x - 2}$ cont at $x = 2$?

Sol: $f(2) = \frac{0}{0}$ undefined.

$\Rightarrow f$ is discontinous.

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim \frac{(x+2)(x-2)}{(x-2)} = 4 \text{ (exists)}.$$

ex: is $X=0$ a removable discontinuity of $f(x) = x^2 \sin(\frac{1}{x})$?.

Sol: $f(0) =$ undefined.

$\lim_{x \to 0} f(x) = 0$ (حسب سؤال السابق) exists by Sandwich Thm.

$\Rightarrow f$ is discontinous at $x=0$

The continous extension of $f(x)$ at $X=0$ is:

$$F(x) = \begin{cases} f(x), & x \neq 0 \\ \lim_{x \to 0} f(x), & x = 0 \end{cases}$$

$$= \begin{cases} x^2 \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

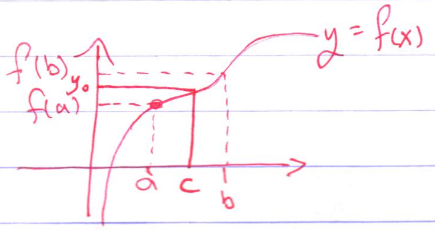
ex: find $F(x)$ of

$f(x) = \frac{x^2-4}{x-2}$ at $x=2$

Sol: $F(x) \begin{cases} \frac{x^2-4}{x-2} & \text{at } x=2. \\ 4, & x=2. \end{cases}$

* Theorem (The intermediate Value Theorem). (IVT)

if f is continous on $[a,b]$ "closed interval", and if y is any Value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ where $c \in [a,b]$.



NOTE: If $y_0 = 0$, $f(c) = 0$
$\Longrightarrow$ f has a route in $[a,b]$.

Q5 P12: ex: Show that $f(x) = x^3 - 2x^2 + 2$ has a root?

Sol: $f(0) = 2$ (positive).
$f(-1) = -1$ (negative).

1: $f(x) = x^3 - 2x^2 + 2$ is continous on $[-1,0]$ Since it is polynomial.

2: $f(-1)$, $f(0)$ have different sign

3: IMVT has a root in $[-1,0]$.

2.2.1: Asymptotes:

* دائماً إذا كانت قوة البسط أقل من قوة المقام فإن الـ H.A يساوي صفر.

Definition: (Horizontal Asymptotes) (H.A)

$$\lim_{x\to\infty} f(x) = b, \text{ and } \lim_{x\to-\infty} f(x) = c$$

$\Rightarrow y = b$ and $y = c$ are H.A's.

ex: find horizontal .A of $f(x) = \dfrac{x^2}{5x^2+1}$

Sol: $\displaystyle\lim_{x\to\infty} \frac{x^2}{5x^2+1} = \frac{1}{5}$

نفس قوة البسط = قوة المقام

$y = \frac{1}{5}$ is H.A.

ex: $f(x) = \dfrac{\sin x}{x}$, find H.A

Sol: $\displaystyle\lim_{x\to\infty} \frac{\sin x}{x} = 0$, $\displaystyle\lim_{x\to-\infty} \frac{\sin x}{x} = 0$ $\rightarrow$ $y = 0$, is H.A

$y = \dfrac{5}{x+3} + 3$

$\downarrow$ H.A $= 3$

ex: $f(x) = \dfrac{5x}{x^2+1}$

$$\lim_{x\to\pm\infty} \frac{5x}{x^2+1} = \lim_{x\to\pm\infty} \frac{5x}{x^2\left(1+\frac{1}{x^2}\right)}$$

$$= \lim_{x\to\pm\infty} \frac{\frac{5}{x}}{1+\frac{1}{x^2}} = \frac{0}{1+0} = 0$$

$y = 0$, is H.A

ex: $f(x) = \cos\left(\frac{1}{x}\right)$

$\lim_{x \to \pm\infty} \cos\left(\frac{1}{x}\right) = 1 \Rightarrow y = 1$, is H.A.

ex: $f(x) = e^{-x}$

$\lim_{x \to +\infty} e^{-x} = 0 \Rightarrow y = 0$, is H.A.

$\lim_{x \to -\infty} e^{-x} = e^{\infty} = \infty \longrightarrow X$

ex: $f(x) = x^2$

$\lim_{x \to \infty} x^2 = \infty$, $\lim_{x \to -\infty} x^2 = \infty$

$\Rightarrow$ there is no H.A.

Vertical Asymptotes (V.A)

$x = a$ is V.A if

$\lim_{x \to a^+} f(x) = \infty$ or $\lim_{x \to a^-} = \infty$

or $\lim_{x \to a^+} f(x) = -\infty$ or $\lim_{x \to a^-} = -\infty$

ex: find the V.A and H.A.

of $f(x) = \dfrac{x-2}{x^2-4}$

Solution: H.A $\lim_{x \to \pm\infty} \dfrac{x-2}{x^2-4} = 0$

$\Rightarrow y = 0$, is H.A

V.A, $x = 2$ / $\lim_{x \to 2^+} \dfrac{x-2}{x^2-4} = \dfrac{1}{4}$

is neither $+\infty$ nor $-\infty$

$\Rightarrow x = 2$ is not V.A (hole).

$x = -2$, $\lim_{x \to -2^+} \dfrac{\cancel{x-2}}{\cancel{(x-2)}(x+2)} = \infty$

$\Rightarrow x = -2$ is V.A

* Oblique (slant) Asymptote (O.A)

قسمة مطولة

$q(x) = \dfrac{p(x)}{r(x)}$ / degree of $p(x)$ is 1 Greater degree of $r(x)$.

ex: find all asymptotes of:

Q_2, ©: $f(x) = \dfrac{x^2+1}{x-1}$

$$
\begin{array}{r|l}
 & x+1 \\
\hline
x-1 & x^2+1 \\
 & x^2-x \\
\hline
 & x+1 \\
 & -x+1 \\
\hline
 & 2
\end{array}
$$

نقلب الإشارة

$\Rightarrow y = x+1$ is O.A.

- Horizontal (H.A) $= \lim\limits_{x \to \infty} \dfrac{x^2+1}{x-1}$

$= \lim\limits_{x \to \infty} \dfrac{x^2(1+\frac{1}{x^2})}{x(1-\frac{1}{x})}$

$= \infty \cdot \left(\dfrac{1+0}{1-0}\right) = \infty$

$\lim\limits_{x \to -\infty} = -\infty$

V.A , $X=1$??

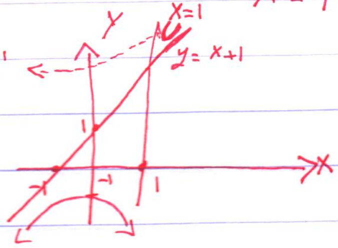
$$\lim_{x \to 1} \frac{x^2+1}{x-1}$$

عندما يكون عدد حلل 0 يكون V وعندما تكون درجة البسط أكبر من درجة المقام يكون O.

ex: sketch $f(x) = \dfrac{x^2+1}{x-1}$

- Asymptotes: $y = x+1$ is O.A.
  $x = 1$ is V.A.

الرسم هنا سنكونه



ex: $f(x) = \dfrac{x^3+1}{x^2}$

find all asymptotes of sketch:

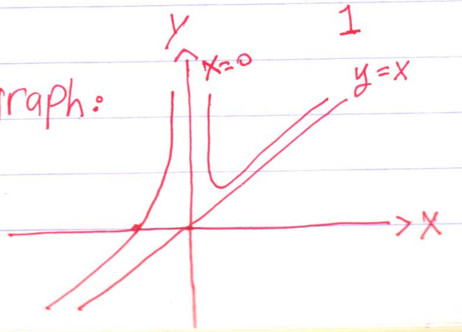
Sol: H.A ✗

V.A $x = 0$

O.A

$$\begin{array}{r|l} & x \\ \hline x^2 & x^3+1 \\ & -x^3 \\ \hline & 1 \end{array} \longrightarrow y = x$$

graph:



Q_2 : a:) $f(x) = \frac{x+1}{x-1}$

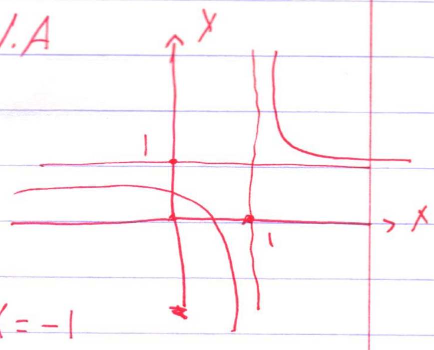
Sol: There's no O.A

H.A : $\lim_{x \to \mp\infty} \frac{x+1}{x-1} = \frac{1}{1} = 1$

$\boxed{y=1}$, is H.A

V.A : $\lim_{x \to 1^+} \frac{x+1}{x-1} = +\infty$

$\boxed{x=1}$ is V.A



X- Intercept

$\frac{x+1}{x-1} = 0 \Rightarrow x = -1$

y-intercept, $x=0 \Rightarrow y=-1$.

CH2

Q1: $$\lim_{t \to -1} \frac{t^2+3t+2}{t^2-t-2}$$ D

$$\lim_{t \to -1} \frac{\cancel{(t+1)}(t+2)}{\cancel{(t+1)}(t-2)}$$

$$= \frac{-1}{3}$$

b: $$\lim_{x \to 1} \frac{1-\sqrt{x}}{1-x}$$

$$= \lim_{x \to 1} \frac{\cancel{1-\sqrt{x}}}{\cancel{(1-\sqrt{x})}(1+\sqrt{x})} = \frac{1}{2}$$

c: $$\lim_{\theta \to 1} \frac{\theta^4-1}{\theta^3-1} = \lim_{\theta \to 1} \frac{(\theta^2-1)(\theta^2+1)}{(\theta-1)(\theta^2+\theta+1)}$$

$$= \lim_{\theta \to 1} \frac{\cancel{(\theta-1)}(\theta+1)(\theta^2+1)}{\cancel{(\theta-1)}(\theta^2+\theta+1)}$$

$$= \frac{2 \times 2}{3} = \frac{4}{3}.$$

d: $$\lim_{\theta \to 0} \frac{\sin 2\theta}{3\theta} = \lim_{\theta \to 0} \frac{\frac{\sin 2\theta}{\cancel{\theta}}}{\frac{3\theta}{\cancel{\theta}}} = \frac{2}{3}$$

* $$\lim_{\theta \to 0} \frac{\sin a\theta}{\theta} = \underline{\underline{a}}$$

$$e{:})\ \lim_{\theta \to 0} \frac{1-\cos\theta}{\sin 2\theta} \cdot \frac{(1+\cos\theta)}{(1+\cos\theta)}$$

D

$$= \lim_{\theta \to 0} = \frac{(1-\cos^2\theta)}{2\sin\theta\cos\theta(1+\cos\theta)}$$

$$\lim_{\theta \to 0} \frac{\sin^2\theta}{2\sin\theta\cos\theta\ (1+\cos\theta)}$$

$$\lim_{\theta \to 0} \frac{\sin\theta}{2\cos\theta\ (1+\cos\theta)} = \frac{0}{2\times 1\times 2} = 0$$

$$f{:})\ \lim_{x \to +\infty} \frac{1+\sqrt{x}}{1-\sqrt{x}} = -1$$

$\sqrt{x^2} = |x|$

$$g{:})\ \lim_{x \to -\infty} \frac{\sqrt{x^2+1}}{x+1} = \lim_{x \to -\infty} \frac{|x|\sqrt{1+\frac{1}{x^2}}}{x+1}$$

$$= -1$$

homework

Q1: H, K

Q2: C, D

$$i{:})\ \lim_{x \to \infty} \sqrt{x^2+1} - \sqrt{x^2-x}$$

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$$= \lim_{x \to \infty} \frac{x^2+1-x^2+x}{\sqrt{x^2+1}+\sqrt{x^2-x}}$$

$$= \lim_{x \to \infty} \frac{x+1}{|x|(\sqrt{1+\frac{1}{x^2}} + \sqrt{1-\frac{1}{x}})} = 1$$

odd $f(-x) = -fx$
Even $f(-x) = fx$

Q2 : a:) $f(x) = \dfrac{X+1}{X-1}$

$$\lim_{x \to \pm\infty} \frac{X+1}{X-1} = 1$$

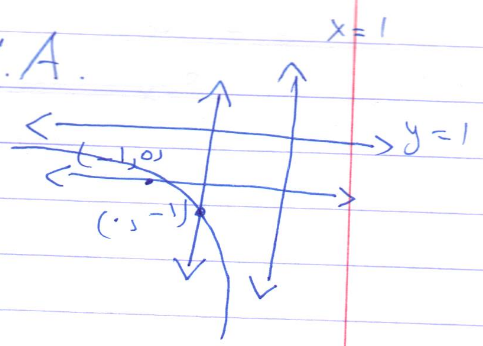
* oblique ← $y = 1$ is the horizontal asymptote

$$\lim_{x \to 1^+} \frac{X+1}{X-1} = \infty \quad , \quad \lim_{x \to -1} \frac{X+1}{X-1} = -\infty$$

$X = 1$ is V.A.

$X = 0 \ / \ y = -1$

$y = 0 \ / \ X = -1$



$$* \ y = \frac{X^3 + 1}{X^2}$$

$$= X + \frac{1}{X^2}$$

$$\lim_{x \to \infty} y = \infty$$

$$\lim_{x \to -\infty} y = -\infty$$

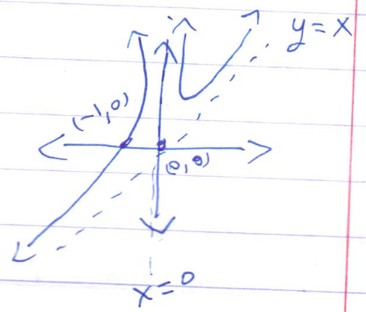
$f(x) = X$ is the oblique Asymptote.

$$\lim_{x \to 0^+} \frac{x^5+1}{x^2} = \infty$$

$$\lim_{x \to 0^-} \frac{x^3+1}{x^2} = \infty$$

$x=0$ is the V.A

$y=0$ / $x=-1$



(102.01) TT

CH2:

Q4: find all continuous extensions of the functions:

$$h(t) = \frac{t^2+3t-10}{t-2}$$

$$\lim_{t \to 2} \frac{t^2+8t-10}{t-2}$$

$$\lim_{t \to 2} \frac{\cancel{(t-2)}(t+5)}{\cancel{(t-2)}} = 7$$

$$h(t) = \begin{cases} \dfrac{t^2+3t-10}{t-2} , & t \neq 2 \\ 7 , & t = 2 \end{cases}$$

Q5: use the I.V.T to show that $f(x) = x^5 - 2x^2 + 2$ has a root.

$f(x)$ is continuous, sin Polynomial.

$f(1) = +1 - 2 + 2 = \cancel{3}\,1 \cancel{>} 0$

$f(-1) = -1 - 2 + 2 = -1 < 0$

$\exists\, c \in (-1, 1)$ s.t $f(c) = 0$