CH4 Applications of derivatives:

[4.1] Increasing and decreasing functions:

definition 4.1.1: let $f(x)$ be a function defined on $\underset{\text{interval}}{I}$:

ⓐ: f is increasing on I. if whenever $x_1 > x_2$ then $f(x_1) > f(x_2)$. $x_1, x_2 \in I$.

ⓑ: f is decreasing on I if $x_2 > x_2$ then $f(x_2) < f(x_1)$, $x_1, x_2 \in I$.

$\nearrow \searrow$

Monotonic

Theorem 4.1.1, Suppose that f is continuous on $[a,b]$ and diffable on (a,b) then:

ⓐ: if $f'(x) \underset{\forall x \in (a,b)}{>0}$ then f is increasing on $[a,b]$

ⓑ: if $f'(x) < 0$. $\forall x \in (a,b)$ then f is decreasing on $[a,b]$.

ex: let $f(x) = x^3 - 12x - 5$

determine the Interval wher f is increasing or decreasing.

Sol: $f'(x) = 3x^2 - 12 = 0$

$\Rightarrow x^2 - 4 = 0$

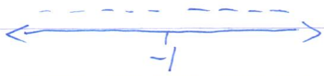
$\Rightarrow x = \pm 2$

then f is increasing on $(-\infty, -2] \cup [2, \infty)$.
and f is decreasing on $[-2, 2]$.

ex: $y = 1 - (x+1)^3$

$y' = -3(x+1)^2 (1) = 0$

$\boxed{x = -1}$



f is decreasing on $\mathbb{R}$.

* notice that f is continuous at $x = -1$ since it's polynomial [إذا لم تكن متصلة يصبح الجواب on R-{-1}].

[4.2] : Extreme Values of functions

Definition 4.2.1 : let f be a function with domain D, then :

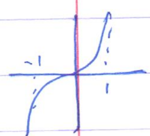
a: f has an absolute maximum on D at $\underline{c}$, if $f(x) < f(c)$, $\forall x \in D$.

b: f has an absolute minimum Value on D at a Point a if $f(x) \geq f(a)$ $\forall x \in D$.

Remark: $f(c)$ is local maximum if the inequality in a holds in a Small Interval around $X = C$. Similarly for b but it's called local minimum.

every absolute is local but not every local is absolute...

ex: find the absolute maximum and absolute minimum of $f(x) = x^3$ on $[-1, 1]$.



* Absolute max $= f(1) = 1$.
* Absolute min $= f(-1) = -1$.

Theory 4.2.1: If f is continuous on a closed Interval $[a, b]$, then f has both an abs max and abs min.

ex: $f(x) = x^3$ is cont at $[-1,1]$.
then f has abs max and abs min according to the previous theory.

~~xxxxx~~ Definition 4.2.2: An <u>interior point</u> in the domain where $f' = 0$ or $f' = DNE$ is called a <u>critical point</u> of f.

ex: find the critical point of
$f(x) = x^{\frac{2}{3}}$. $[-1, 8]$

Domain $= [-1, 8]$ given.
f is continuous on $[-1, 8]$

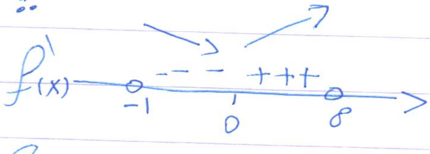
$\frac{2}{3} x^{-\frac{1}{3}}$

$= \frac{2}{3x^{\frac{1}{3}}}$ → $f' = 0$ when $2 = 0$ X
→ DNE when $3x^{\frac{1}{3}} = 0$ $\therefore$ $\underline{\underline{x = 0}}$

* <u>End points</u> $x = -1$ / $x = -8$

النقاط الطرفية ليس نقاط حرجة

$\therefore$



f is decreasing on $[-1, 0]$
f is increasing on $[0, 8]$

$f(0) = 0$ abs min

4.2. continue

Theorem 4.2.2: if f is differentiable and has an extreme value at an interior point $x = c$ the $\underline{\underline{f'(c) = 0}}$.

(العكس)

The converse is not true in general

$$f'(c) = 0 \not\Longrightarrow f \text{ has extreme values.}$$

"لا يعني دائماً"

ex: $f(x) = x^3$

$f'(x) = 3x^2$

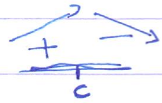
$f'(0) = 0$ but f has no extreme values at $x = 0$.

النقاط الحرجة
ليست نقاط حدية

* 1st derivative test:

Suppose that f has critical point $x = c$,

ex: * f' changes from + to − then $f(c)$ is local max.



* f' changes from − to + then $f(c)$ is local minimum.

* if f' doesn't change then there's no extreme values.

ex: find the intervals in which $f(x) = x^4 - 2x^2$ is ↗ or ↘ find the extreme values

Sol: $x^4 - 2x^2$

* "دائماً" أول خطوة في تحديد المجال Domain.

Domain $= (-\infty, \infty)$.

f is continuous and differentiable on $(-\infty, \infty)$.

$f'(x) = 4x^3 - 4x = 0$

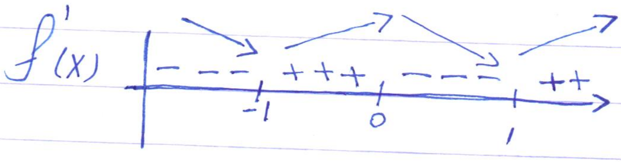
$4x(x^2 - 1) = 0$

$4x(x-1)(x+1)$

$x = 0$, $x = 1$, $x = -1$

theres no end point:

the critical points $\{0, -1, 1\}$.



* f is increasing on $[-1, 0] \cup [1, \infty)$.
* f is decreasing on $(-\infty, -1] \cup [0, 1]$.

$f(-1) = (-1)^4 - 2(-1)^2 = -1$ local minimum value.

$f(0) = 0 - 0 = 0$ local maximum value.

$f(1) = 1-2 = -1$ (local minimum Value).

$f(-1) = -1$ and $f(1) = -1$ ...
both are absolute minimum Values.

*2nd derivative test:

$f'(x) = 0$
$x = c$
$f''(c) = 0$

* إذا فشل اختبار المشتقة الثانية نرجع إلى المشتقة الأولى وهو سيفشل فقط عندما المشتقة الثانية تساوي صفر.

$f''(c) > 0$
f has local minimum at $x = c$

$f''(c) = 0$
test here fails and we go back to 1st test

$f''(c) < 0$
f has local maximum at $x = c$

ex: use second derivative test to find the extreme values of $f(x) = x^4 - 2x^2$

Sol: • $f'(x) = 4x^3 - 4x$
• $f''(x) = 12x^2 - 4$

• $f'(x) = 0 \Rightarrow x = 1,\ x = 0,\ x = -1$.

• $f''(1) = 12 - 4 > 0$ $\therefore f(1)$ is a local min Value.
• $f''(0) = -4 < 0$ $\therefore f(0)$ is a local maximum Value.
• $f''(-1) = 12 - 4 > 0$ $\therefore f(-1)$ is a local min Value.

Concavity: التقعر:

* if $f''(x) > 0$ for all $x \in I$, then f is concave **up** on I.

* if $f''(x) < 0$ for all $x \in I$, then f is concave **down** on I.

* a point where f has tangent line and changes concavity is called an inflection point.

ex: find the Intervals at which the function $f(x) = x^4 - 4x^3 + 10$ is ~~max~~ concave up and down, determine the inflection points (if any)?

$$f(x) = x^4 - 4x^3 + 10$$
$$f'(x) = 4x^3 - 12x^2$$
$$f''(x) = 12x^2 - 24x$$

*ملاحظة: في التقعر لا يوجد فترات مغلقة لذلك كلها تكتب على شكل فترات ( ).

$f''(x) = 0$, $12x^2 - 24x = 0$, $12x(x-2) = 0$

$\Rightarrow \boxed{x=0}$, $\boxed{x=2}$

+ + + + + | 0 | − − − − | 2 | + + + +

∪ ∩ ∪

f is concave up on: $(-\infty, 0) \cup (2, \infty)$.

" " " down on: $(0, 2)$.

inflection points are: $(0, f(0))$, $(2, f(2))$

$(0, 10)$, $(2, -6)$.

Since f is continuous there and the concavity changes there.

Curve sketching :

ex: Sketch $f(x) = \frac{x^2-3}{x-2}$

①: Domain : $(-\infty, 2) \cup (2, \infty)$.

②: Intercepts : x-inter $\Rightarrow y=0 \Rightarrow x^2-3=0$, $x = \pm\sqrt{3}$.

y-inter $\Rightarrow x=0 \Rightarrow y = \frac{3}{2}$.

③: H.A ∝ $\left(\lim_{x \to \pm\infty} f(x) = \infty \right.$

V.A $x=2$ ✓ $\lim_{x \to 2\pm} \frac{x^2-3}{x-2} = \pm\infty$

O.A

ex: let $f(x) = \frac{x^2}{x^2-1}$, $f'(x) = \frac{-2x}{(x^2-1)^2}$ (check)

$f''(x) = \frac{6x^2+2}{(x^2-1)^3}$, find the following:

①: Domain of f.

Df: $x^2-1 \neq 0 \Rightarrow x \neq 1^{\pm}$

D: $(-\infty,-1) \cup (-1,1) \cup (1,\infty)$

②: Asymptotes:

H.A, $\lim_{x \to \pm\infty} \frac{x^2}{x^2-1} = 1$, $y=1$ is H.A.

V.A. $x=1$, $x=-1$

Since $x \to 1^+$ $\frac{x^2}{x^2-1} = \infty$

$\lim_{x \to -1} \frac{x^2}{x^2-1} = -\infty$

There's no O.A

③: Intercepts:

x. inter: $y=0 \to x^2=0 \Rightarrow \underline{x=0}$

y. inter: $x=0 \to y = \frac{0}{0-1} = \underline{y=0}$

④: Critical points:

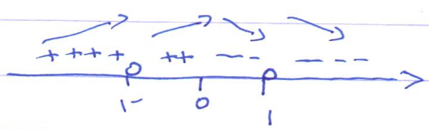
$f' = 0$ or f' DNE

$$f'(x) = \frac{-2x}{(x^2-1)^2} = 0$$

$-2x = 0 \quad \therefore \underline{\underline{x = 0}}$

$x^2 - 1 = 0 \quad x = \pm 1$, they are not in the domain, so there's only one critical point $= 0$.
$= (0, f(0))$

$f'(x)$



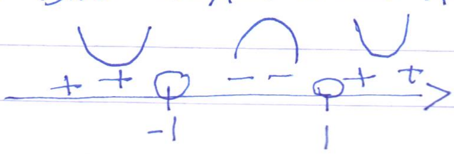
f is $\nearrow$ on $(-\infty, -1) \cup (-1, 0]$
f is $\searrow$ on $[0, 1) \cup (1, \infty)$

$f(0) = 0$ is local max and absolute max Value.

⑤: Concavity: $f''(x) = \dfrac{6x^2+2}{(x^2-1)^3}$

$f'' = 0 \Rightarrow 6x^2 + 2 = 0 \quad \emptyset$,

$f''(x)$ DNE $\Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$.



* هذا المثال يعتبر ربع إمتحان first.

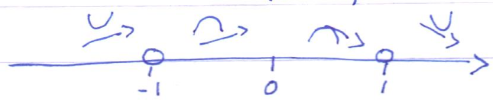
f is concave up on $(-\infty,-1) \cup (1,\infty)$

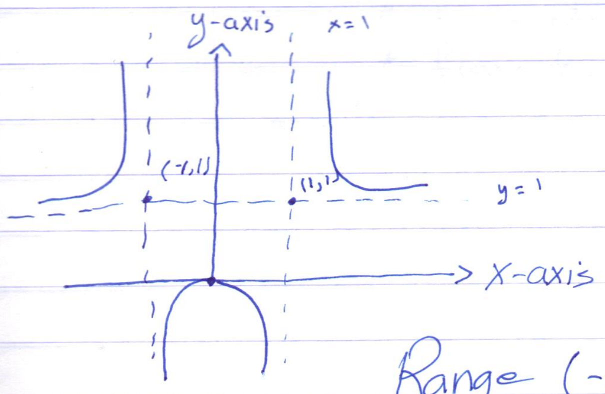
" " " down on $(-1,1)$

There's no inflection points

(Notice that the concavity changes about $x = \pm 1$ but f is discont there).

⑥: sketch $y = f(x) = \dfrac{x^2}{x^2-1}$

f' and f'' 



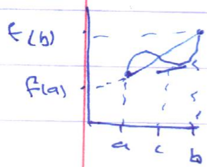
* نرسم من صورة الرقم وليس من الرقم نفسه

Range $(-\infty,0] \cup (1,\infty)$.

4.3: the mean Value thm

THM (MVT) if $y = f(x)$ is continuous on $[a, b]$ and is diffable on (a, b) then there is atleast $c \in (a, b)$:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Thm (Rolle's):

if: * f is cont on $[a, b]$
* f is diffable on (a, b)
* $f(a) = f(b)$

$\Rightarrow$ there's $c \in (a, b)$: $f'(c) = 0$

ex: $Q_2/22$ find c in the conclusion of the mean value theorem for $f(x) = \sqrt{x}$ on $[1, 4]$.

Sol: * f is cont on $[1, 4]$
* $f'(x) = \frac{1}{2\sqrt{x}}$ is diffable on $[1, 4]$.

by MVT THM, $f'(c) = \frac{f(4) - f(1)}{4 - 1}$

$$\frac{1}{2\sqrt{c}} = \frac{2 - 1}{3} \Rightarrow 2\sqrt{c} = 3$$

$c = \frac{9}{4}$, $\in (1, 4)$.

ex. $f(x) = \sqrt{x}$, $[a, b]$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\frac{1}{2\sqrt{c}} = \frac{\sqrt{b} - \sqrt{a}}{b - a} \cdot \frac{\sqrt{b} + \sqrt{a}}{\sqrt{b} + \sqrt{a}}$$

$$= \frac{1}{2\sqrt{c}} = \frac{1}{\sqrt{b} + \sqrt{a}}$$

$$\Rightarrow c = \left(\frac{\sqrt{b} + \sqrt{a}}{2}\right)^2 \in (a, b).$$