*CH5: Integration

integral sign $\int f(x)\,dx$: the integral of $f(x)\,dx$

integrand

Def, F is called antiderivative of $f(x)$ if $F'(x) = f(x)$ or $\int f(x)\,dx = F(x) + c$

ex: • $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$, $n \neq -1$

* $\int x^5\,dx = \frac{x^6}{6} + c$

* $\int \sin x\,dx = -\cos x + c$
* $\int \cos x\,dx = \sin x + c$
* $\int \sec^2 x\,dx = \tan x + c$
* $\int \sec x \tan x\,dx = \sec x + c$
* $\int \csc x \cot x\,dx = -\csc x + c$
* $\int \csc^2 dx = -\cot x + c$

ex: $\int \sin^2 x\,dx = \int \frac{1-\cos(2x)}{2}\,dx$

$= \frac{1}{2}\int [1-\cos(2x)]\,dx$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + c$

ex $\int \sqrt{x^4 + x^{-4} + 2}\,dx$

ex: $\int \sqrt{x^4+x^{-4}+2}\,dx$

$= \int \sqrt{(x^2+x^{-2})^2}\,dx$

$= \int (x^2+x^{-2})\,dx$

$= \int x^2 dx + \int x^{-2} dx$

$= \frac{x^3}{3} + \frac{x^{-1}}{-1} + c$.

ex: $\int \tan^2 x\,dx = \int (\sec^2 - 1)\,dx$

$= \tan x - x + c$.

* EXERCISES: Q_1, , Q_4, Q_5

Q_1:

a:) $\int \sin(5x)\,dx = \frac{-\cos(5x)}{5} + c$

b:)

c:) $\int (1+\cot^2\theta)\,d\theta$

$= \int \csc^2\theta\,d\theta = -\cot\theta + c$.

d:) $\int \frac{\csc\theta\,d\theta}{\csc\theta - \sin\theta} = \int \frac{\frac{1}{\sin\theta}}{\frac{1}{\sin\theta} - \frac{\sin\theta}{1}}\,d\theta$

$$= \int \frac{1}{1-\sin^2\theta}\, d\theta \quad = \int \frac{1}{\cos^2\theta}\, d\theta$$

$$= \int \sec^2\theta \, d\theta \; = \tan\theta + c$$

Q4, a:) $\int_1^{\sqrt{2}} \frac{x^2+\sqrt{x}}{x}\, dx$

$$= \int_1^{\sqrt{2}} (x + x^{-\frac{1}{2}})\, dx = \left(\frac{x^2}{2} + 2\sqrt{x}\right)\Big|_1^{\sqrt{2}}$$

$$= \left[\frac{(\sqrt{2})^2}{2} + 2\sqrt{2}\right] - \left[\frac{1}{2} + 2\right]$$

$$= -\frac{3}{2} + 2\sqrt[4]{2}$$

b:) $\int_0^{\frac{\pi}{6}} (\sec x + \tan x)^2\, dx$

$$\int_0^{\frac{\pi}{6}} (\sec^2 x + 2\sec x \tan x + \underbrace{\tan^2 x}_{\sec^2 x - 1})\, dx$$

$$\int_0^{\frac{\pi}{6}} (2\sec^2 x + 2\sec x \tan x - 1)\, dx$$

$$= (2\tan x + 2\sec x - x)\Big|_0^{\frac{\pi}{6}}$$

$$= \left[2\tan\frac{\pi}{6} + 2\sec\frac{\pi}{6} - \frac{\pi}{6}\right] - [2]$$

$$= 2 \cdot \frac{1}{\sqrt{3}} + 2 \cdot \frac{2}{\sqrt{3}} - \frac{\pi}{6} - 2$$

$$= \frac{6}{\sqrt{3}} - \frac{\pi}{6} - 2$$

c: $\int_0^{\pi} [\cos x + |\cos x|] \, dx$

$= \int_0^{\frac{\pi}{2}} (\cos x + \cos x) \, dx + \int_{2\pi}^{\pi} (\cancel{\cos x - \cos x}) \, dx$

$= \int_0^{\frac{\pi}{2}} 2\cos x \, dx = 2\sin x \Big|_0^{\frac{\pi}{2}}$

$= 2(1) - 2(0) = 2$

* INTEGRATION by Subtitution:

Q5: a:) $I = \int \frac{1}{\sqrt{x}(1+\sqrt{x})^2} \, dx$

let $u = \sqrt{x} + 1$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$dx = 2\sqrt{x} \, du$

$\therefore I = \int \frac{1}{\sqrt{x} u^2} \cdot 2\sqrt{x} \, du$

$= 2\int u^{-2} \, du$

$= \frac{-2}{u} + C$

$= \frac{-2}{1+\sqrt{x}} + C$

b:)

$$I = \int \frac{\sec x \tan x}{\sqrt{\sec x}}\, dx$$

let $u = \sqrt{\sec x}$

$$\frac{du}{dx} = \frac{\sec x \tan x}{2\sqrt{\sec x}}$$

$$dx = \frac{2\sqrt{\sec x}}{\sec x \tan x}\, du$$

$$I = \int \frac{\cancel{\sec x \tan x}}{\cancel{u}} \cdot \frac{2\cancel{\sqrt{\sec x}}}{\cancel{\sec x \tan x}}\, du$$

$$\therefore I = \int 2\, du = 2u + c = 2\sqrt{\sec x} + c.$$

d: $I = \int x^3 \sqrt{x^2+1}$

let $u = \sqrt{x^2+1}$

$$\frac{du}{dx} = \frac{x}{\sqrt{x^2+1}} \Rightarrow dx = \frac{u}{x}\, du$$

$$dx = \frac{\sqrt{x^2+1}}{x}\, du$$

$$I = \int x^3 u \frac{\sqrt{x^2+1}}{x}\, du = \int x^2 u . u\, du$$

$$= \int (u^2-1)\, u^2 \, du$$

$u = \sqrt{x^2+1}$

$\therefore x^2 = u^2 - 1$

$$\int (u^4 - u^2)\, du$$

$$\frac{1}{5} u^5 - \frac{u^3}{3} + c$$

$$= \frac{1}{5}\left(\sqrt{x^2+1}\right)^5 - \frac{\left(\sqrt{x^2+1}\right)^3}{3} + c$$

* Definite Integrals and Area

Fundemental Theorem of calculas.

① f is cont on $[a,b]$ and $\int f(x)\,dx = F(x) + c$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

* $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$

In general, $F(x) = \int_{g(x)}^{h(x)} f(t)\,d\cancel{x}t$

$$F'(x) = f(h(x))\,h'(x) - f(g(x))\,g'(x)$$

ex: $F(x) = \int_1^{x^2} \frac{1}{1+t^3}\, dt$, find $F'(x)$

Sol: $F'(x) = \frac{1}{1+(x^2)^3} \cdot 2x - 0$

$= \frac{2x}{1+x^6}$

ex: $\frac{d}{dx}\int_0^x \frac{\sin t}{t}\, dt = \frac{\sin x}{x} \cdot 1 - 0$

$= \frac{\sin x}{x}$

ex: $F(x) = \int_{x^3}^{\sin x} \frac{1}{1+t^2}\, dt$

$F'(x) = \frac{1}{1+\sin^2 x} \cdot \cos x - \frac{1}{1+x^6} \cdot 3x^2$

$= \frac{\cos x}{1+\sin^2 x} - \frac{3x^2}{1+x^6}$

ex: $y = x^2 \cdot \int_1^x \frac{\sin t}{t^2+4}\, dt$

$y' = 2x \int_1^x \frac{\sin t}{t^2+4}\, dt + x^2\left(\frac{\sin x}{x^2+4}\right).$

Q3 : find the linearization of

$$f(x) = 3 + \int_1^{x^2} \sec(t-1)\,dt$$

at $\boxed{x=-1}$

sol. $a = -1$

$$f(-1) = 3 + \cancel{\int_1^1 \sec(t-1)\,dt}^{\,0} = 3$$

$$f(-1) = 3$$

$$f'(x) = 0 + \sec(x^2-1) \cdot 2x - 0$$

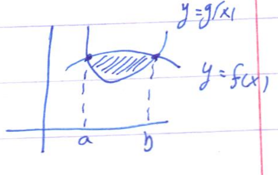
$$f'(-1) = \sec^2(1-1)(2)(-1) = -2$$

$$L(x) = f(-1) + f'(-1)(x+1)$$

$$= 3 + -2(x+1)$$

$$l(x) = -2x + 1$$

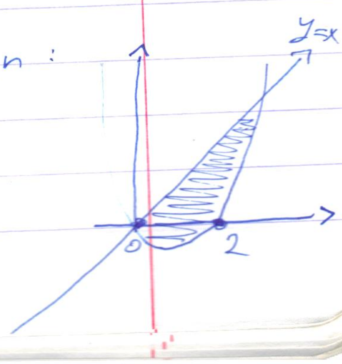
*AREA:



$$\text{area} = \int_a^b [f(x) - g(x)]\,dx$$

Q6 : find the area between :

a.) $y = x^2 - 2x$, $y = x$



$$x^2 - 2x = 0$$

$$x(x-2) = 0 \ / \ x = \underline{0, 2}$$

*Intersection points:

$x^2 - 2x = \cancel{x}$

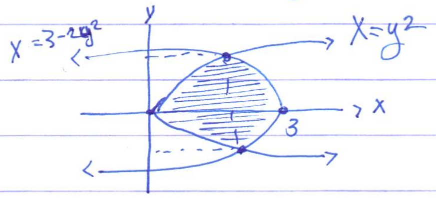
$x(x-3) = 0$

$x = 0 \;, x = 3$

area: $\int_0^3 x - (x^2 - 2\cancel{x}] \, dx$

$= \int_0^3 (\bar{3}x - x^2) \, dx$ .....

(c:) $x = y^2 \,,\, x = 3 - 2y^2$



*Intersection: $3 - 2y^2 = y^2$

$y = \pm 1$

Area $= \int_{-1}^{1} (3 - 2y^2 - y^2) \, dy$

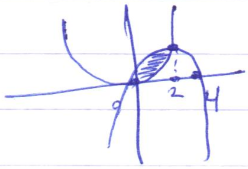
$= \int_{-1}^{1} (3 - 3y^2) \, dy$

$= 3y - y^3 \Big|_{-1}^{1}$

$= (3-1) - (-3+1)$

$= 2 + 2 = \underline{\underline{4}}$

B:) $y = x^2$, $y = -x^2 + 4x$



$-x^2 + 4x = 0$

$x(-x + 4) = 0$

$x = 0$, $x = 4$

Intersection point : $-x^2 + 4x = x^2$

$\Rightarrow 2x^2 - 4x = 0 \Rightarrow 2x(x-2) = 0$

$\boxed{x = 0} \Rightarrow \boxed{x = 2}$

$$\text{Area} = \int_0^2 [(-x^2 + 4x) - (x^2)]\, dx$$

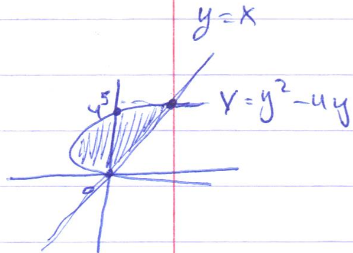
$$= \int_0^2 (4x - 2x^2)\, dx$$

$$= 2x^2 - \frac{2x^3}{3} \Big|_0^2$$

$$= 8 - \frac{16}{3} = \frac{8}{3}$$

ex: $x = y^2 - 4y$, $x = y$

$x = 0$, $y^2 - 4y = 0$, $y = 0$, $y = 4$



Intersections : $y^2 - 4y = y$

$y^2 - 5y = 0$

$y = 0$, $y = 5$

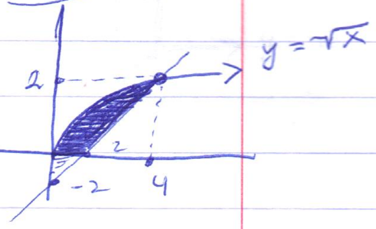
$$\therefore \text{Area} = \int_0^5 y - (y^2 - 4y)\, dy$$

$$= \int_0^5 (5y - y^2)\, dy$$

$x = y^2$

* example: $y = \sqrt{x}$, x-axis

$y = x - 2$ $\rightarrow$ $x = y + 2$



intersections: $x - 2 = \sqrt{x}$

$x^2 - 4x + 4 = x$

$x^2 - 5x + 4 = 0$

$(x-1)(x-4) = 0$

$x = \{1, 4\}$

rejected

* method 1: x-axis

$$\text{Area} = \int_0^2 (\sqrt{x} - 0)\,dx + \int_2^4 \left[\sqrt{x} - (x-2)\right] dx$$

* method 2: y-axis

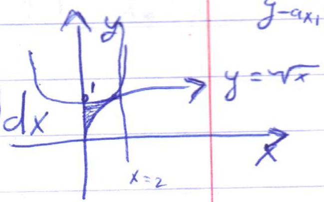
$$\text{Area} = \int_0^2 (y+2) - y^2\,dy$$

$$= \frac{y^2}{2} + 2y - \frac{y^3}{3} \Big|_0^2$$

$$2 + 4 - \frac{8}{3} = \frac{10}{3}$$

*example: $y = x^2+1$, $y = \sqrt{x}$, $x = 2$, $x = 0$
y-axis

$$Area = \int_0^2 (x^2+1-\sqrt{x})\,dx$$



$\sqrt{x} =$ 

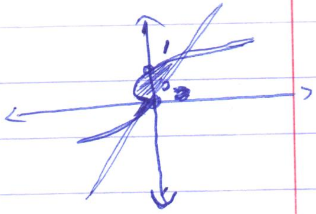
Q6: d:1

$x = y^3 - y^2$, $x = 2y$

$x = 0$ $y^3 = y^2 = 0$

$y^2(y-1) = 0$

$\boxed{y=0}$ $\boxed{y=1}$



Intersections

$y^3 - y^2 = 2y$

$y(y^2-y-2) = 0 \Rightarrow y(y-2)(y+1) = 0$

$\boxed{y=0}$, $\boxed{y=2}$, $\boxed{y=-1}$

$$*\ Area = \int_{-1}^{0} (y^3-y^2-2y)\,dy + \int_0^2 (2y-y^3+y^2)\,dy.$$

$$I = \int \sqrt{\frac{x^3-1}{x^{11}}} \implies = \int \sqrt{\frac{x^3-1}{x^3 \, x^8}}$$

$$dx = \int \frac{1}{x^4} \sqrt{\frac{x^3-1}{x^3}} \, dx$$

$$= \int x^{-4} \sqrt{1-x^{-3}} \, dx$$

let $u = 1-x^{-3} \implies \frac{du}{dx} = 3x^{-4}$

$$dx = \frac{du}{3x^{-4}}$$

$$I = \int x^{-4} \sqrt{4} \, \frac{du}{3x^{-4}} = \frac{1}{3} \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{2}{9}(1-x^{-3})^{\frac{3}{2}} + c$$

ex: $\int \frac{1}{x^{2018}-x} \, dx$

$$= \int \frac{1}{x^{2018}} \cdot \frac{1}{1-x^{-2017}} \, dx$$

$$= \int \frac{x^{-2018}}{1-x^{-2017}} \, dx$$, let $y = 1-x^{-2017}$

$$\frac{dy}{dx} = 2017x^{-2018}$$

$$dx = \frac{dy}{2017x^{-2018}}$$

$$= \frac{1}{2017}\int \frac{1}{y}\,dy = \frac{1}{2017}\ln|y| + c$$

$$\frac{1}{2017}\ln|1-x^{-2017}| + c$$

$$*\ ex: \int \frac{1}{\sqrt{\cos^3 x \sin x}}\,dx$$

$$= \int \frac{1}{\sqrt{\frac{\cos^4 x}{\cos x}\sin x}}\,dx$$

$$= \int \frac{1}{\cos^2 x\sqrt{\tan x}}\,dx$$

$$= \int \frac{\sec^2 x}{\sqrt{\tan x}}\,dx \qquad \text{let } y = \tan x$$

$$\frac{dy}{dx} = \sec^2 x$$

$$\frac{\cancel{\sec^2 x}}{\sqrt{y}}\ \frac{dy}{\cancel{\sec^2 x}} \qquad dx = \frac{dy}{\sec^2 x}$$

$$= 2\sqrt{y} + c = 2\sqrt{\tan x} + c.$$