

Review of Differentiation and Integration (ch1-ch5)

Chapter 1: Functions: (1.1)

Def: a function f is a rule that assigns to each point x in the domain D a unique point $y = f(x)$ in the range of f , R

$$f: D \rightarrow R$$

D : Domain $المجال$

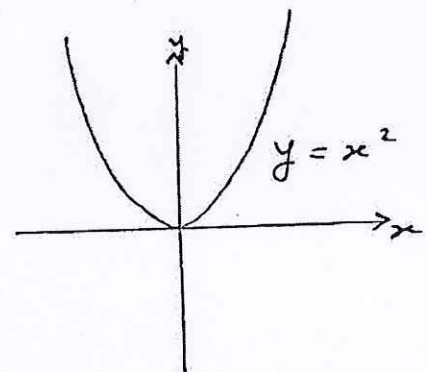
R : range $المدى$

Example: Find D and R for

Ⓐ $f(x) = x^2$.

Domain = $(-\infty, \infty) = \mathbb{R}$

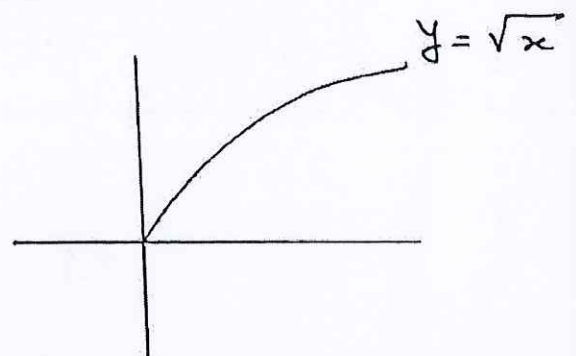
Range = $[0, \infty)$.



Ⓑ $f(x) = \sqrt{x}$

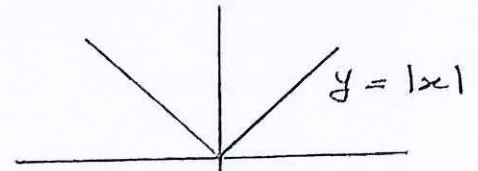
Domain = $[0, \infty)$

Range = $[0, \infty)$



③ $f(x) = |x| =$ absolute value of x .

$$f(x) = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$$



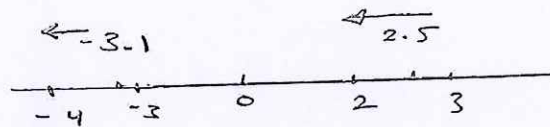
Domain = $(-\infty, \infty) = \mathbb{R}$ (The set of all real numbers)

Range = $[0, \infty)$

④ The greatest integer function

$$\lfloor x \rfloor = \begin{cases} \vdots & \\ 1 & , 1 \leq x < 2 \\ 0 & , 0 \leq x < 1 \\ -1 & , -1 \leq x < 0 \\ \vdots & \end{cases}$$

$$\lfloor 2.5 \rfloor = 2$$

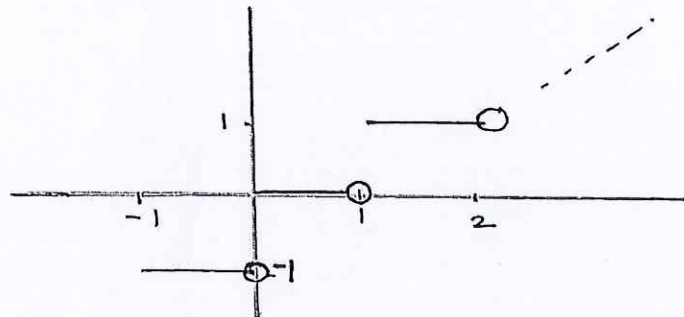


$$\lfloor -3.1 \rfloor = -4$$

Domain = $(-\infty, \infty) = \mathbb{R}$

Range = $\{0, \pm 1, \pm 2, \dots\}$

$\mathbb{R} =$ set of Integers.



$$\textcircled{c} \quad y = f(x) = \lfloor |x| \rfloor$$

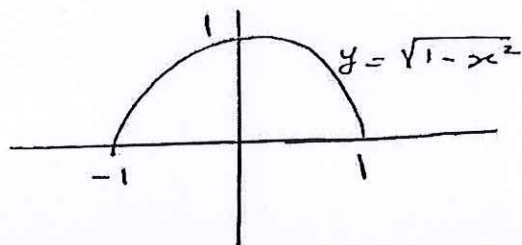
$$\text{Domain} = (-\infty, \infty)$$

$$\text{Range} = \{0, 1, 2, 3, \dots\} = \text{The set of positive Integers Union \{Zero\}.$$

$$\textcircled{d} \quad y = f(x) = \sqrt{1-x^2}$$

$$y = \sqrt{1-x^2}$$

$$y^2 = 1-x^2 \Leftrightarrow x^2+y^2=1$$



$$\text{Domain} = [-1, 1]$$

$$\text{Range} = [0, 1]$$

$$\text{(For the domain, we need } 1-x^2 \geq 0 \Leftrightarrow 1 \geq x^2$$

$$\Leftrightarrow 1 \geq \sqrt{x^2} \Leftrightarrow 1 \geq |x| \Leftrightarrow 1 \geq x \geq -1.)$$

$$\textcircled{e} \quad y = f(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\text{Domain} = (-1, 1) \quad [\text{since } 1-x^2 > 0]$$

$$\text{Range} = [1, \infty)$$

1.2 Trigonometric functions

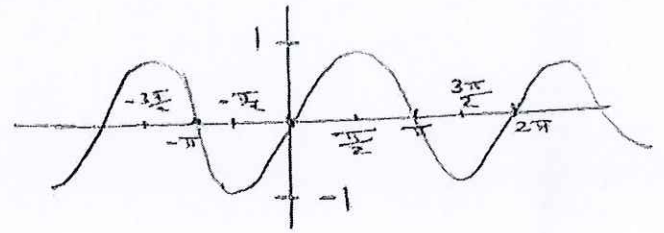
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① $y = \sin x$,

"O-A=OP"

Domain = $(-\infty, \infty)$

Range (R) = $[-1, 1]$

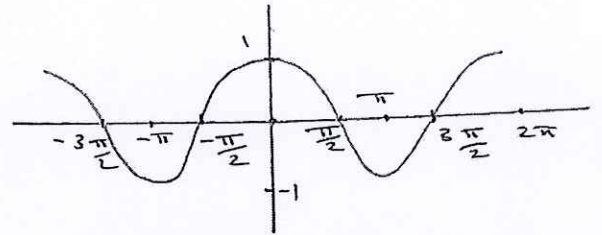


② $y = \cos x$,

"O-A=OP"

Domain = $(-\infty, \infty)$

Range = $[-1, 1]$



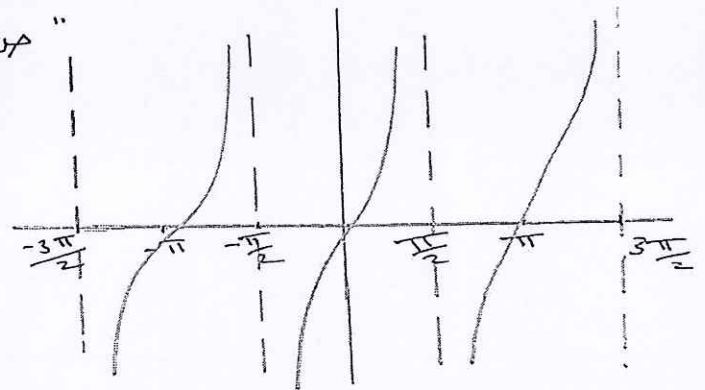
③ $y = \tan x$,

"O-A=OP"

$= \frac{\sin x}{\cos x}$

Domain: $(-\infty, \infty) \setminus \left\{ \frac{\pi}{2} \pm n\pi \right\}$
where $n = 0, 1, 2, \dots$

Range: $(-\infty, \infty)$



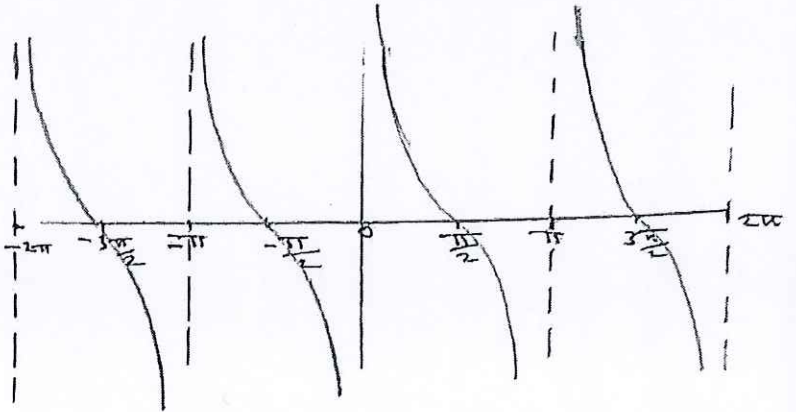
④ $y = \cot x$, "0-ly" = up"

$$= \frac{\cos x}{\sin x}$$

Domain = $(-\infty, \infty) \setminus \{\pm n\pi\}$

where $n = 0, 1, 2, \dots$

Range = $(-\infty, \infty)$



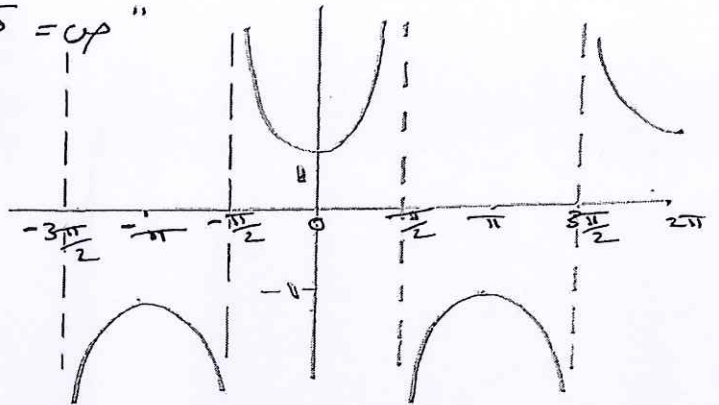
⑤ $y = \sec x$, "0-ly" = up"

$$= \frac{1}{\cos x}$$

Domain = $(-\infty, \infty) \setminus \{\frac{\pi}{2} \pm n\pi\}$

where $n = 0, 1, 2, \dots$

Range = $(-\infty, -1] \cup [1, \infty)$



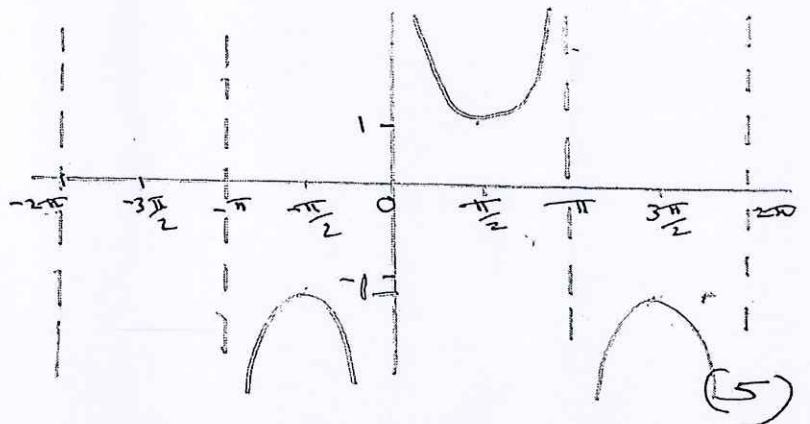
⑥ $y = \csc x$, "0-ly" = up"

$$= \frac{1}{\sin x}$$

Domain = $(-\infty, \infty) \setminus \{\pm n\pi\}$

where $n = 0, 1, 2, \dots$

Range = $(-\infty, -1] \cup [1, \infty)$



Remark ①

$$(1) \sin(x + 2\pi) = \sin x$$

$$(2) \cos(x + 2\pi) = \cos x$$

$$(3) \sec(x + 2\pi) = \sec x$$

$$(4) \csc(x + 2\pi) = \csc x$$

} periodic functions
with
period 2π .

while:

$$(5) \tan(x + \pi) = \tan x$$

$$(6) \cot(x + \pi) = \cot x$$

} period π

Trigonometric Identities:

$$(1) \sin^2 x + \cos^2 x = 1$$

$$(2) \sin(2x) = 2 \sin x \cos x$$

$$(3) \cos(2x) = \cos^2 x - \sin^2 x$$

$$(4) \cos^2 x = \frac{1 + \cos(2x)}{2}$$

$$(5) \sin^2 x = \frac{1 - \cos(2x)}{2}$$

(6)

$$(6) \quad \sec^2 x = 1 + \tan^2 x$$

$$(7) \quad \csc^2 x = 1 + \cot^2 x$$

$$(8) \quad \cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$(9) \quad \sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$(10) \quad \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Example: (a) $\sin(x+\pi) = \sin x \overset{=-1}{\cos \pi} + \cos x \overset{=0}{\sin \pi}$
 $= -\sin x$

(b) $\sin(2x) = \overset{A+B}{\sin(x+x)} = \sin x \cos x + \cos x \sin x = 2 \sin x \cos x.$

(c) $\cos(2x) = \overset{A+B}{\cos(x+x)} = \cos x \cos x - \sin x \sin x$
 $= \cos^2 x - \sin^2 x.$

(d) $\cos\left(x + \frac{\pi}{2}\right) = \cos x \overset{=0}{\cos \frac{\pi}{2}} - \sin x \overset{=1}{\sin \frac{\pi}{2}}$
 $= \cos x (0) - \sin x (1) = -\sin x$

1.3 Even and Odd Functions:

Def: A function $y = f(x)$ is an:

(1) even function if $f(-x) = f(x)$

Its graph is symmetric about y -axis.

(2) odd function if $f(-x) = -f(x)$

Its graph is symmetric about the origin.

Example: Determine whether the following functions are even, odd or neither.

(1) $f(x) = x^2 + 1$

$$f(-x) = (-x)^2 + 1 = x^2 + 1 = f(x) \Rightarrow \begin{array}{l} f \text{ is} \\ \text{Even} \\ \text{function} \end{array}$$

(2) $f(x) = x^3 + x$

$$\begin{aligned} f(-x) &= (-x)^3 + (-x) = -x^3 - x \\ &= -(x^3 + x) = -f(x) \end{aligned}$$

Therefore f is an odd function.

$$(3) \quad f(x) = x + 1$$

$$f(-x) = -x + 1 \neq f(x)$$

and $\neq -f(x)$

Therefore $f(x)$ is Neither

$$(4) \quad f(x) = |x|$$

$$f(-x) = |-x| = |x| = f(x) \Rightarrow f \text{ is even}$$

$$(5) \quad f(x) = \sin x$$

$$f(-x) = \sin(-x) = \sin(0-x) = \overset{0}{\sin} \cos x - \overset{1}{\cos} \sin x$$
$$= -\sin x = -f(x) \Rightarrow f \text{ is odd.}$$

Remark (2)

$$y = \sin x$$

$$y = \tan x$$

$$y = \cot x$$

$$y = \csc x$$

} Odd functions.

while

$$y = \cos x$$

$$y = \sec x$$

} Even functions.

Chapter 2 : Limits and Continuity :

2.1 Limits of functions :

$$\lim_{x \rightarrow a} f(x) = L$$

"J = (a) to L"

means : $f(x)$ goes $\rightarrow L$ as x goes $\rightarrow a$

"The function f gets arbitrarily close to L as x is sufficiently close to a "

Note: $\lim_{x \rightarrow a} f(x) = L \iff \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L$
right hand limit \quad left hand limit

Example: Find the following limits :

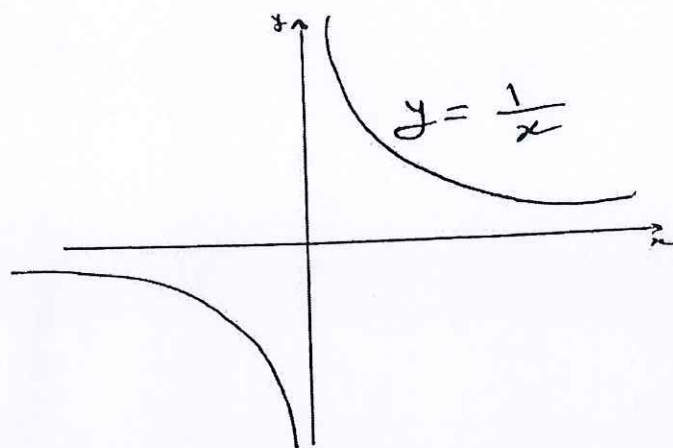
$$\textcircled{1} \lim_{x \rightarrow 1} \frac{x-1}{x+1} = \frac{1-1}{1+1} = \frac{0}{2} = 0 \quad (\text{exists})$$

This means that $f(x) = \frac{x-1}{x+1} \rightarrow 0$
as $x \rightarrow 1$.

$$\textcircled{2} \quad \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \frac{1 - 1}{1 - 1} = \frac{0}{0}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{(x-1)(x+1)}{(x-1)} = \lim_{x \rightarrow 1} (x+1) = 2$$

$$\textcircled{3} \quad \lim_{x \rightarrow \infty} \frac{1}{x} = 0$$



$$\textcircled{4} \quad \lim_{x \rightarrow -\infty} \frac{1}{x} = 0$$

$$\textcircled{5} \quad \lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$$

$$\textcircled{6} \quad \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$$

$$\textcircled{7} \quad \lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x} = \frac{0}{0} = \lim_{x \rightarrow 1} \frac{(x-1)(x+2)}{x(x-1)}$$

$$= \lim_{x \rightarrow 1} \frac{x+2}{x} = \frac{1+2}{1} = 3 \quad (\text{exists})$$

$$\textcircled{8} \quad \lim_{x \rightarrow -1} \frac{\sqrt{x^2+8} - 3}{x+1} = \frac{\sqrt{9} - 3}{-1+1} = \frac{0}{0}$$

$$= \lim_{x \rightarrow -1} \left(\frac{\sqrt{x^2+8} - 3}{x+1} \right) \cdot \left(\frac{\sqrt{x^2+8} + 3}{\sqrt{x^2+8} + 3} \right) \quad \text{"الضرب بالمرافقة"}$$

$$= \lim_{x \rightarrow -1} \frac{\overbrace{(\sqrt{x^2+8})^2}^{\text{فرصه بين مربعين}} - (3)^2}{(x+1)(\sqrt{x^2+8} + 3)} = \lim_{x \rightarrow -1} \frac{(x^2+8) - 9}{(x+1)(\sqrt{x^2+8} + 3)}$$

$$= \lim_{x \rightarrow -1} \frac{x^2 - 1}{(x+1)(\sqrt{x^2+8} + 3)} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow -1} \frac{(x-1)(\cancel{x+1})}{(x+1)(\sqrt{x^2+8} + 3)}$$

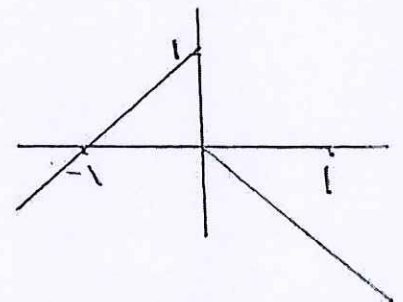
$$= \frac{-1-1}{\sqrt{1+8} + 3} = \frac{-2}{\sqrt{9} + 3} = \frac{-2}{3+3} = \frac{-2}{6} = -\frac{1}{3}$$

$$\textcircled{9} \quad \lim_{x \rightarrow 0} f(x), \text{ where } f(x) = \begin{cases} x+1, & x \leq 0 \\ -x, & x > 0. \end{cases}$$

$$\cdot \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (-x) = \boxed{0}$$

$$\cdot \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (x+1) = 0+1 = \boxed{1}$$

$$\Rightarrow \lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x) \Rightarrow \lim_{x \rightarrow 0} f(x)$$



Does NOT Exist
(DNE)

(12)

(10) $\lim_{x \rightarrow -1} g(x)$ and $\lim_{x \rightarrow 1} g(x)$

where $g(x) = \begin{cases} x+2 & , x \leq -1 \\ x^2 & , -1 < x \leq 1 \\ x-1 & , x > 1 \end{cases}$

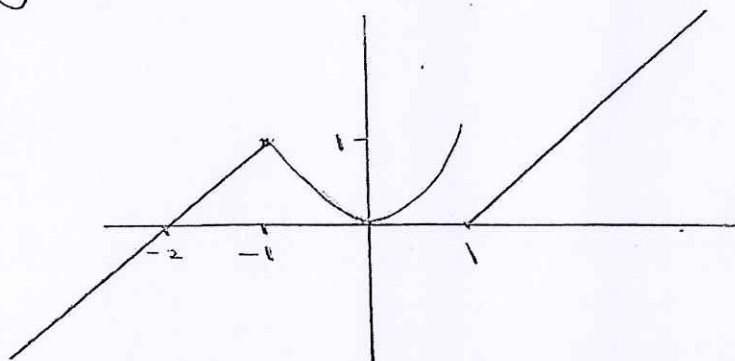
$\lim_{x \rightarrow -1^+} g(x) = (-1)^2 = 1$, $\lim_{x \rightarrow -1^-} g(x) = -1 + 2 = 1$

so $\lim_{x \rightarrow -1} g(x) = \square$

while: $\lim_{x \rightarrow 1^+} g(x) = 1 - 1 = 0$, $\lim_{x \rightarrow 1^-} g(x) = (1)^2 = 1$

$\Rightarrow \lim_{x \rightarrow 1^+} g(x) \neq \lim_{x \rightarrow 1^-} g(x)$

$\Rightarrow \lim_{x \rightarrow 1} g(x)$ Does Not Exist. (DNE).



graph of $g(x)$.

(13)

Theorem: (The Sandwich theorem): (Squeeze thm.)

Suppose that $g(x) \leq f(x) \leq h(x)$

for all x in some open interval containing c ,

except at $x = c$, and

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L, \text{ then:}$$

$$\lim_{x \rightarrow c} f(x) = L.$$

Example: Suppose that $\underbrace{1-x^2}_{g(x)} \leq f(x) \leq \underbrace{1+x^2}_{h(x)}$.

Find $\lim_{x \rightarrow 0} f(x)$.

$$\lim_{x \rightarrow 0} \overset{g(x)}{(1-x^2)} = 1-0=0, \quad \lim_{x \rightarrow 0} \overset{h(x)}{(1+x^2)} = 1+0=0$$

$\Rightarrow \lim_{x \rightarrow 0} f(x) = 1$ by Sandwich theorem.

Example: If $2 - x^2 \leq g(x) \leq 2 \cos x$, $\forall x$

Find $\lim_{x \rightarrow 0} g(x)$

$$\lim_{x \rightarrow 0} (2 - x^2) = 2, \quad \lim_{x \rightarrow 0} 2 \cos x = 2 \cos(0) = 2 \cdot 1 = 2$$

$\Rightarrow \lim_{x \rightarrow 0} g(x) = 2$ by Sandwich theorem.

Example: Find $\lim_{x \rightarrow +\infty} \frac{\sin x}{x}$.

We know that $-1 \leq \sin x \leq 1$

$$\Rightarrow \frac{-1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

$$\lim_{x \rightarrow +\infty} \frac{-1}{x} \leq \lim_{x \rightarrow +\infty} \frac{\sin x}{x} \leq \lim_{x \rightarrow +\infty} \frac{1}{x}$$

$$0 \leq \lim_{x \rightarrow +\infty} \frac{\sin x}{x} \leq 0$$

Therefore: $\lim_{x \rightarrow +\infty} \frac{\sin x}{x} = 0$

Similarly: $\lim_{x \rightarrow +\infty} \frac{\cos x}{x} = 0$ (by the same reason) above

Remark ①: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

Example: Find $\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right)$

We know: $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

$$\lim_{x \rightarrow 0} (-x^2) = \lim_{x \rightarrow 0} (x^2) = 0 \Rightarrow \lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$

Example: $\lim_{x \rightarrow 0} \sqrt{x^4 + x^2} \sin\left(\frac{\pi}{x}\right)$

$$-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1$$

$$-\sqrt{x^2 + x^4} \leq \sqrt{x^4 + x^2} \sin\left(\frac{\pi}{x}\right) \leq \sqrt{x^4 + x^2}$$

$$\lim_{x \rightarrow 0} (-\sqrt{x^4 + x^2}) = \lim_{x \rightarrow 0} (\sqrt{x^4 + x^2}) = \sqrt{0+0} = \sqrt{0} = 0.$$

$$\Rightarrow \lim_{x \rightarrow 0} \sqrt{x^4 + x^2} \sin\left(\frac{\pi}{x}\right) = 0 \quad \text{by Sandwich thm.}$$

(16)

2.2 Continuity

Def: A function f is continuous at $x = a$ if

(1) $f(a)$ exists (موجود)

(2) $\lim_{x \rightarrow a} f(x)$ exists

(3) $\lim_{x \rightarrow a} f(x) = f(a)$

Examples: ① $y = \sin x$, $y = \cos x$, $y = |x|$ are continuous on $(-\infty, \infty)$.

② polynomials (كثيرات الحدود) are continuous on $(-\infty, \infty)$
ex: $f(x) = x^4 + 5x^3 - 2x + 1$

③ $g(x) = \frac{x^3 + x - 5}{x^2 - 1}$ is continuous on $\mathbb{R} \setminus \{-1, 1\}$
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④ $h(x) = \frac{x^2 - 5}{x^2 + 3x}$ is continuous on $\mathbb{R} \setminus \{0, -3\}$

Example: (A function with removable discontinuity).

$$f(x) = \frac{x^2 + 2x - 3}{x^2 - 1} \quad \text{is continuous on } \mathbb{R} \setminus \{1, -1\}$$

$$\frac{x^2 + 2x - 3}{x^2 - 1} = \frac{(x-1)(x+3)}{(x-1)(x+1)}$$

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Note that $f(1)$ is undefined "غير معرف"

but at $x \rightarrow 1$, $\lim_{x \rightarrow 1} f(x) = \lim_{\substack{x \rightarrow 1 \\ x \neq 1}} \frac{\cancel{(x-1)}(x+3)}{\cancel{(x-1)}(x+1)} = \frac{4}{2} = \boxed{2}$

In this case $x=1$ is called a removable discontinuity.

We can define a continuous extension of f at $x=1$

as :

$$F(x) = \begin{cases} f(x), & x \neq 1 \\ \lim_{x \rightarrow 1} f(x), & x = 1 \end{cases} = \begin{cases} f(x), & x \neq 1 \\ 2, & x = 1 \end{cases}$$

$$= \begin{cases} \frac{x^2 + 2x - 3}{x^2 - 1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$$

Example: Find the Continuous extension $F(x)$ of

$$f(x) = \frac{x^2 + 3x - 10}{x - 2}.$$

$$f(x) = \frac{x^2 + 3x - 10}{x - 2} = \frac{(x - 2)(x + 5)}{(x - 2)}$$

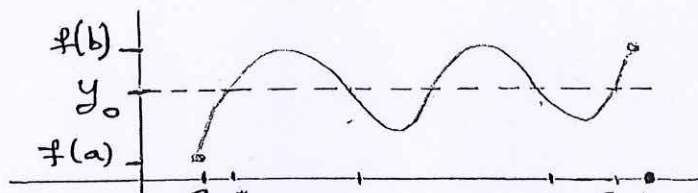
$$\Rightarrow \lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} (x + 5) = 7$$

$$\Rightarrow F(x) = \begin{cases} \frac{x^2 + 3x - 10}{x - 2}, & x \neq 2 \\ 7, & x = 2. \end{cases}$$

$x = 2$ is called a removable discontinuity.

Theorem: The Intermediate Value Theorem. (IMVT)

If f is continuous function on $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $\exists c \in [a, b]$ such that $f(c) = y_0$.



Note: If $y_0 = 0$ in IMVT,

$f(c) = y_0 = 0$, then c is called a root of f .
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(i.e) : since $y_0 = 0$ is between $f(a)$ and $f(b)$,

then $f(a) < 0 < f(b)$ ($\frac{0-f(a)}{f(b)-f(a)}$)
المقايمة

this means $f(a) \cdot f(b) < 0$. (تغير الإشارة)

Theorem: Bolzano Theorem

If f is continuous on $[a, b]$ and if $f(a) \cdot f(b) < 0$

then there exists $c \in [a, b]$ such that $f(c) = 0$.

Example: show that $f(x) = x^3 - x - 1$ has a root in $[1, 2]$.

Using IMVT (in this case Bolzano theorem):

1) $f(x)$ is continuous on $[1, 2]$ since it's polynomial.

2) $f(1) = 1^3 - 1 - 1 = -1 < 0$, $f(2) = 2^3 - 2 - 1 = 5 > 0$

then: $f(1) < 0 < f(2)$, that is $f(1)f(2) < 0$,

so by IMVT (Bolzano theorem): $\exists c \in [1, 2]$

such that $f(c) = 0$, (i.e) f has a root on $[1, 2]$. (20)

Asymptotes: " الخطوط التقريبية "

There are three types of Asymptotes:

- 1) Horizontal Asymptote. (H.A) " أفقي "
- 2) Vertical Asymptote. (V.A) " عمودي "
- 3) Oblique (slant) Asymptote. (O.A) " مائل "

Def: A line $y = b$ is a Horizontal Asymptote

of $y = f(x)$ if either:

$$\lim_{x \rightarrow \infty} f(x) = b \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = b$$

Note: We are dealing with rational functions:

$$f(x) = \frac{P(x)}{q(x)} = \frac{a_m x^m + a_{m-1} x^{m-1} + \dots + a_1 x + a_0}{b_n x^n + b_{n-1} x^{n-1} + \dots + b_1 x + b_0}$$

= $\frac{\text{polynomial of degree } m}{\text{polynomial of degree } n}$

(a) If $m = n$, then $\lim_{x \rightarrow \pm\infty} f(x) = \frac{a_m}{b_n}$

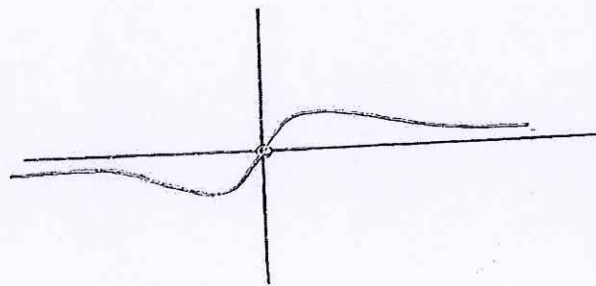
(b) If $m < n$, then $\lim_{x \rightarrow \pm\infty} f(x) = 0$

(c) If $m > n$, then $\lim_{x \rightarrow \pm\infty} f(x) = \pm\infty$.

Example: Find H.A of the graph of $f(x) = \frac{x}{x^2+1}$

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x}{x^2+1} = \lim_{x \rightarrow \infty} \frac{x}{x^2(1+\frac{1}{x})}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{(1+\frac{1}{x})} = 0$$

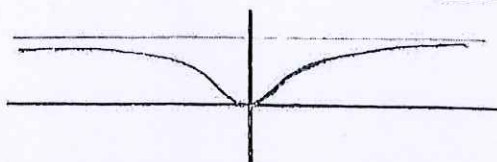


Similarly: $\lim_{x \rightarrow -\infty} f(x) = 0$

\Rightarrow $y=0$ is a Horizontal Asymptote.

Example Find H.A for $f(x) = \frac{2x^2}{x^2+1}$

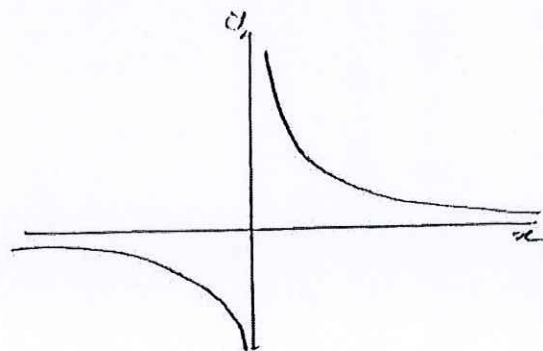
$$\lim_{x \rightarrow \pm\infty} f(x) = \frac{2}{1} = 2 \Rightarrow y=2 \text{ is a H.A}$$



Example: Find H.A for $f(x) = \frac{\sin x}{x^2}$.

$$-1 \leq \sin x \leq 1$$

$$\Rightarrow \frac{-1}{x^2} \leq \frac{\sin x}{x^2} \leq \frac{1}{x^2}$$



Since $\lim_{x \rightarrow \pm\infty} \frac{-1}{x^2} = \lim_{x \rightarrow \pm\infty} \frac{1}{x^2} = 0$, then

by sandwich theorem $\lim_{x \rightarrow \pm\infty} \frac{\sin x}{x^2} = 0$.

$\Rightarrow \boxed{y=0}$ is a H.A for $f(x)$

Example: Find H.A for $f(x) = \frac{\sqrt{x^2+1}}{x}$.

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^2+1}}{x} = \lim_{x \rightarrow \infty} \frac{\sqrt{x^2(1+\frac{1}{x^2})}}{x} \quad \begin{array}{l} |x|=x \\ \text{since } x \rightarrow \infty \end{array}$$

$$= \lim_{x \rightarrow \infty} \frac{\sqrt{x^2} \sqrt{1+\frac{1}{x^2}}}{x} = \lim_{x \rightarrow \infty} \frac{|x| \sqrt{1+\frac{1}{x^2}}}{x}$$

$$= \lim_{x \rightarrow \infty} \frac{x \sqrt{1+\frac{1}{x^2}}}{x} = \lim_{x \rightarrow \infty} \sqrt{1+\frac{1}{x^2}} = \boxed{1} \Rightarrow \boxed{y=1} \text{ is H.A}$$

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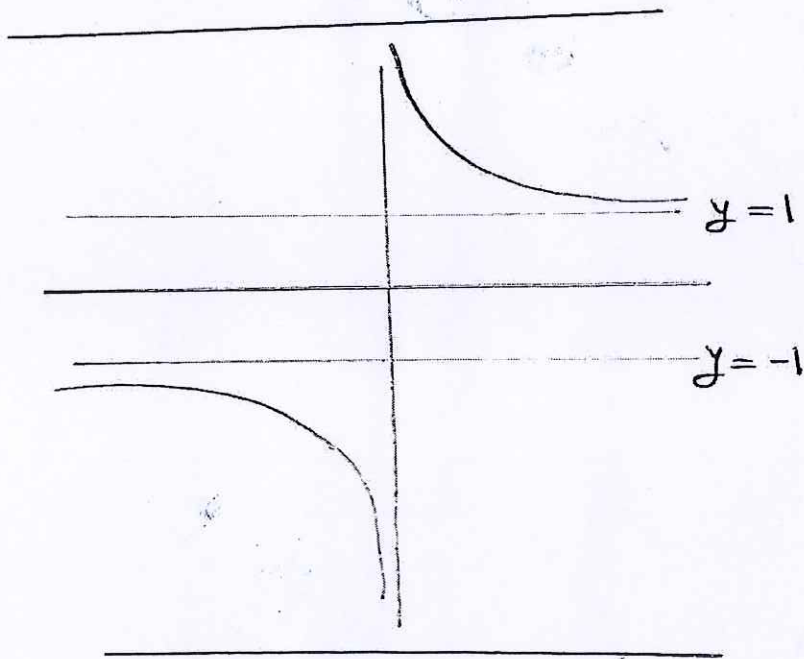
$$\begin{aligned}
 \text{Now: } \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2+1}}{x} &= \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2(1+\frac{1}{x^2})}}{x} \\
 &= \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2} \sqrt{(1+\frac{1}{x^2})}}{x} = \lim_{x \rightarrow -\infty} \frac{|x| \sqrt{1+\frac{1}{x^2}}}{x} \\
 &= \lim_{x \rightarrow -\infty} \frac{\cancel{-x} \sqrt{1+\frac{1}{x^2}}}{\cancel{x}} = \lim_{x \rightarrow -\infty} -\sqrt{1+\frac{1}{x^2}} = \boxed{-1}
 \end{aligned}$$

$|x| = -x$
 since $x \rightarrow -\infty$

then $\boxed{y = -1}$ is a H.A.

Graph of

$$f(x) = \frac{\sqrt{x^2+1}}{x}$$



Def: A Line $x = a$ is a Vertical Asymptote.

If the graph of $y = f(x)$ if either:

$$\lim_{x \rightarrow a^+} f(x) = \pm \infty \quad \text{or} \quad \lim_{x \rightarrow a^-} f(x) = \pm \infty$$

Example: Find the V.A for $y = f(x) = \frac{1}{x}$.

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$$

$$\Rightarrow \boxed{x = 0} \text{ is a V.A.}$$

$$\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$$

Example: Find the V.A for $y = \frac{2x+1}{x-1}$

$$\lim_{x \rightarrow 1^+} \frac{2x+1}{x-1} = +\infty \Rightarrow \boxed{x = 1} \text{ is a V.A.}$$

We can also show that

$$\lim_{x \rightarrow 1^-} \frac{2x+1}{x-1} = -\infty \Rightarrow \boxed{x = 1} \text{ is a V.A.}$$

Example: Find the V.A for $f(x) = \frac{x^2 + 2x - 3}{x^2 - 1}$

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{(x-1)(x+3)}{(x-1)(x+1)} = \frac{4}{2} = \boxed{2} \neq \pm \infty$$

Therefore $\boxed{x=1}$ is NOT a V.A

we call it removable discontinuity.

While:

$$\lim_{x \rightarrow -1^-} \frac{(x-1)(x+3)}{(x-1)(x+1)} = -\infty \Rightarrow \boxed{x=-1} \text{ is a V.A}$$

Example: Find the H.A & V.A for $f(x) = \frac{2x+1}{x-1}$

$$\text{H.A: } \lim_{x \rightarrow \pm 2} \frac{2x+1}{x-1} = \frac{2}{1} = 2 \Rightarrow \boxed{y=2} \text{ is H.A.}$$

$$\text{V.A: } \lim_{x \rightarrow 1^-} \frac{2x+1}{x-1} = -\infty \Rightarrow \boxed{x=1} \text{ is V.A}$$

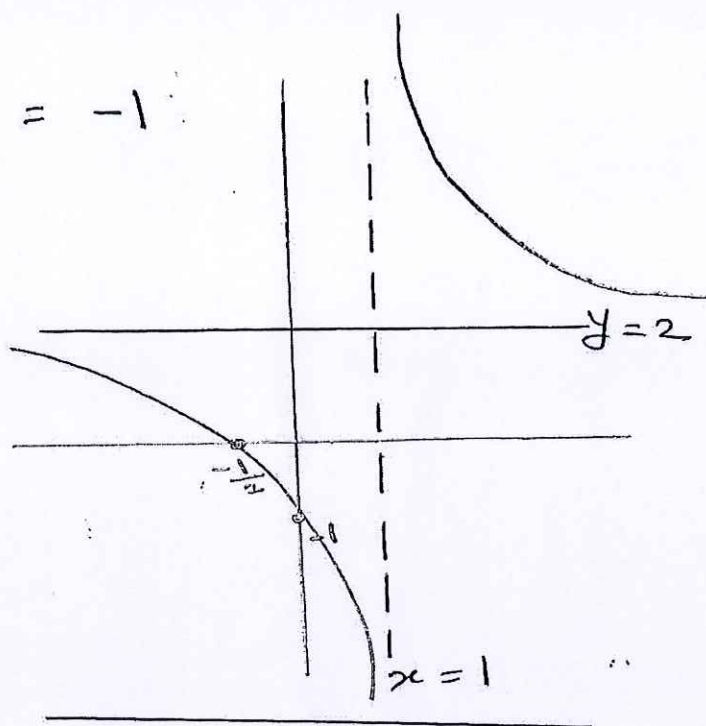
We need to sketch $f(x)$:

x -intercept : Let $y = 0$, then

$$\frac{2x+1}{x-1} = 0 \iff x = -\frac{1}{2}$$

y -intercept : Let $x = 0$, then

$$y = \frac{0+1}{0-1} = -1$$



Remark : Suppose that $f(x)$ is a rational function.

- 1) The graph of $f(x)$ can intersect its H.A.
- 2) The graph of $f(x)$ can have both H.A. & V.A.
- 3) The graph of $f(x)$ have only one H.A.

Remark: $f(x) = \frac{\sin x}{x}$ has **NO** V.A

Since $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$

Def: Oblique Asymptote (O.A):

If the degree of the ^{num} numerator is 1 greater than the degree of the ^{den} denominator, then f has an Oblique Asymptote.

Example: Find the O.A for $f(x) = \frac{x^2}{x-1}$

$$\Rightarrow \frac{x^2}{x-1} = \underbrace{(x+1)}_{\text{O.A}} + \underbrace{\frac{1}{x-1}}_{\substack{0 \\ \text{as } x \rightarrow \infty}}$$
$$\begin{array}{r} x-1 \overline{) \begin{array}{r} x+1 \\ x^2 \\ -x^2+x \\ \hline x \\ -x+1 \\ \hline 1 \end{array}} \end{array}$$

Therefore: $\boxed{y = x+1}$ is O.A

Note that: $\lim_{x \rightarrow 1^-} f(x) = -\infty \Rightarrow \boxed{x=1}$ is V.A

Remark: The rational function cannot have a Horizontal and an Oblique Asymptote at the same time.

Example: Q2) (c) Find the asymptotes of:

$$f(x) = \frac{x^2 + 1}{x - 1}$$

$$\begin{array}{r} x+1 \\ x-1 \overline{) x^2 + 1} \\ \underline{-x^2 + x} \\ x+1 \\ \underline{-x+1} \\ 2 \end{array}$$

$$\Rightarrow f(x) = \underbrace{(x+1)}_{\text{O.A.}} + \frac{2}{x-1}$$

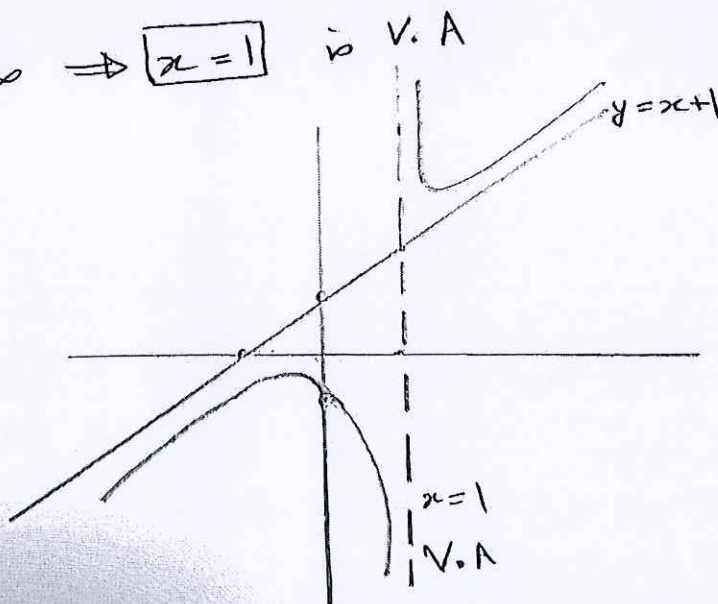
$\Rightarrow \boxed{y = x + 1}$ is the Oblique Asym.

• $\lim_{x \rightarrow \pm\infty} f(x) = \pm\infty \Rightarrow$ There is NO H.A

• $\lim_{x \rightarrow 1^+} f(x) = +\infty \Rightarrow \boxed{x = 1}$ is V.A

• $\lim_{x \rightarrow 1^-} f(x) = -\infty$

y-intercept: $y = -1$



Example Q2) (d)

$$f(x) = \frac{x^3 + 1}{x^2 - 1}$$

$$\frac{x^2 - 1}{x^2 - 1} \sqrt{\frac{x^3 + 1}{x^3 + x}} = \frac{x}{x + 1}$$

$$f(x) = x + \frac{x + 1}{x^2 - 1} \xrightarrow{x \rightarrow \infty} 0$$

\Rightarrow $y = x$ is Oblique Asymptote.

Now:

$$\lim_{x \rightarrow +\infty} f(x) = +\infty \quad \& \quad \lim_{x \rightarrow -\infty} f(x) = -\infty$$

Therefore **(NO)** H.A

$$\text{Now: } f(x) = \frac{x^3 + 1}{x^2 - 1} = \frac{(x + 1)(x^2 - x + 1)}{(x + 1)(x - 1)}$$

$$\lim_{x \rightarrow -1} f(x) = \frac{(-1)^2 - (-1) + 1}{(-1 - 1)} = \frac{3}{-2} \neq \infty \neq -\infty$$

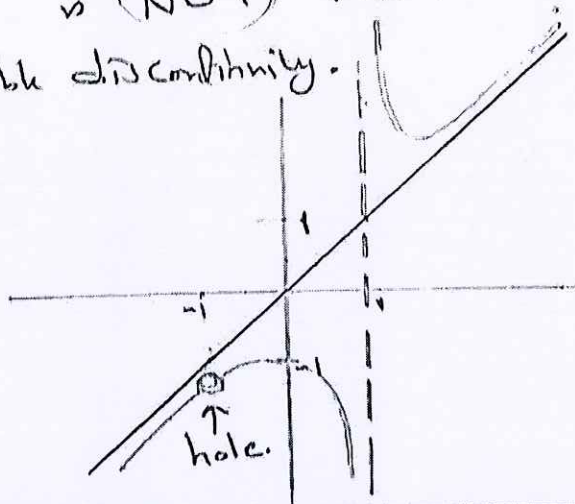
Therefore $x = -1$ is **(NOT)** V.A

We call it removable discontinuity.

while

$$\lim_{x \rightarrow 1^+} f(x) = +\infty$$

\Rightarrow $x = 1$ is V.A



Chapter 3 : Differentiation :

3.1 Definition of derivative:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

(we read it the derivative of f at $x = a$).

• If $f'(a)$ exists, then f is differentiable at $x = a$.

Note: f is differentiable on $[a, b]$ if

(a) f is diff in (a, b)

(b) $f'_+(a) = \lim_{h \rightarrow 0^+} \frac{f(a+h) - f(a)}{h}$ exists. (right hand derivative of f at $x = a$)

(c) $f'_-(b) = \lim_{h \rightarrow 0^-} \frac{f(b+h) - f(b)}{h}$ exists. (left hand derivative of f at $x = b$)

Remark: f is diff. at $x=c$ if and only if

$$f'_+(c) = f'_-(c) \text{ [exist and equal].}$$

Example: Use the definition of derivative to find

$$f'(x) \text{ for } f(x) = \sqrt{x}.$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \rightarrow 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \right) \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right)$$

$$= \lim_{h \rightarrow 0} \frac{(x+h) - (x)}{h(\sqrt{x+h} + \sqrt{x})} =$$

$$= \lim_{h \rightarrow 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} = \frac{1}{2\sqrt{x}}.$$

Theorem: If f is differentiable at $x = c$

then f is continuous at $x = c$

Note: The converse of the theorem is NOT true.

(i.e.) If f is continuous at $x = c$, then

f need not be differentiable at $x = c$.

Example: Let $f(x) = |x|$. $f(x)$ is continuous function at $x = 0$

$$f'_+(0) = \lim_{h \rightarrow 0^+} \frac{|0+h| - |0|}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = 1$$

while:

$$f'_-(0) = \lim_{h \rightarrow 0^-} \frac{|h+0| - |0|}{h} = \lim_{h \rightarrow 0^-} \frac{-h}{h} = -1$$

then $f'_+(0) \neq f'_-(0)$

so f is not differentiable at $x = 0$

Example: Determine whether the following function

is differentiable at $x = 0$:

$$f(x) = \begin{cases} x^{2/3} & , x \geq 0 \\ x^{1/3} & , x < 0. \end{cases}$$

$$\bullet f'_+(0) = \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{h^{2/3}}{h} = \lim_{h \rightarrow 0^+} \frac{1}{h^{1/3}} = +\infty.$$

$$\bullet f'_-(0) = \lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{h^{1/3}}{h} = \lim_{h \rightarrow 0^-} \frac{1}{h^{2/3}} = +\infty$$

So f is Not differentiable at $x = 0$

In this case we say that f has a vertical tangent at $x = 0$.

3.2 Differentiation rules:

Theorem: Suppose $f(x)$ and $g(x)$ are diff. at x ,
and C is constant, then:

$$1) \frac{d}{dx}(c) = 0.$$

$$2) \frac{d}{dx} x^n = n x^{n-1}, \text{ where } n \in \mathbb{Z}^+.$$

$$3) \frac{d}{dx}(c f(x)) = c \frac{d}{dx}(f(x)).$$

$$4) \frac{d}{dx}(f(x) \pm g(x)) = \frac{df}{dx} \pm \frac{dg}{dx}.$$

$$5) \frac{d}{dx}(f(x) \cdot g(x)) = \frac{df}{dx} g(x) + f(x) \frac{dg}{dx}.$$

$$6) \frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{g(x) \frac{df}{dx} - f(x) \frac{dg}{dx}}{g^2(x)}.$$

$$7) \frac{d}{dx}(f \circ g)(x) = \frac{d}{dx} f(g(x)) \cdot \frac{dg(x)}{dx}. \quad (\text{Chain Rule})$$
