8.3 Trigonometric Substitutions:

The most common substitutions are:

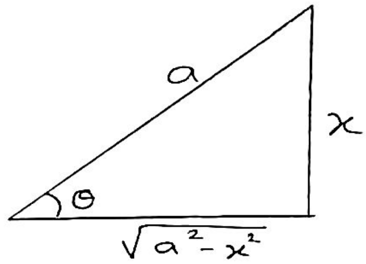
$$x = a\sin\theta \quad , \quad x = a\tan\theta \quad , \quad x = a\sec\theta .$$

These substitutions are efficient in transforming Integrals involving $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$. Into Integrals we can evaluate directly.

1) With $x = a\sin\theta$.

$$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2\cos^2\theta .$$

where $\theta = \sin^{-1}\left(\frac{x}{a}\right)$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$



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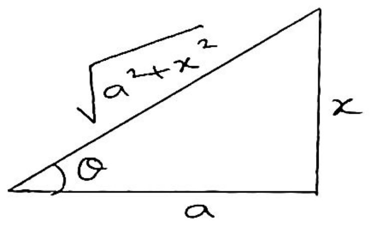
2) With $x = a\tan\theta$

$$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2\sec^2\theta$$

where $\theta = \tan^{-1}\left(\frac{x}{a}\right)$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$



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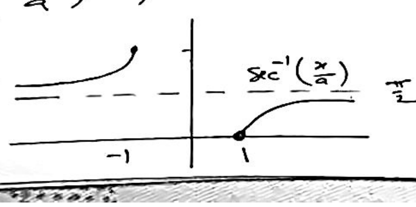
3) With $x = a\sec\theta$

$$x^2 - a^2 = -\left(a^2 - a^2\sec^2\theta\right) = a^2\sec^2\theta - a^2 = a^2\tan^2\theta .$$

where $\theta = \sec^{-1}\left(\frac{x}{a}\right)$, with $\begin{cases} 0 \leq \theta < \frac{\pi}{2} & , \text{if } \frac{x}{a} \geq 1 \\ \frac{\pi}{2} < \theta \leq \pi & , \text{if } \frac{x}{a} \leq -1 \end{cases}$



<u>Example:</u> Evaluate $\int \frac{dx}{\sqrt{4+x^2}}$. $\stackrel{??}{=} \sinh^{-1}\left(\frac{x}{2}\right) + C$.

1) We set $x = 2\tan\theta$, $dx = 2\sec^2\theta\, d\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

2) $4 + x^2 = 4 + 4\tan^2\theta = 4(1+\tan^2\theta) = 4\sec^2\theta$.

3) $\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{\sec^2\theta}{|\sec\theta|}\, d\theta$.

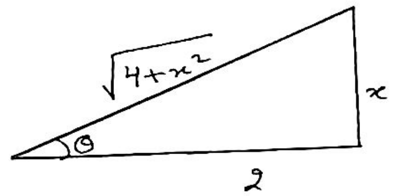
موجبة في الربع الأول والرابع $\frac{1}{\cos\theta} = \sec\theta > 0$

on $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$= \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$

from step 2

4) $= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C$

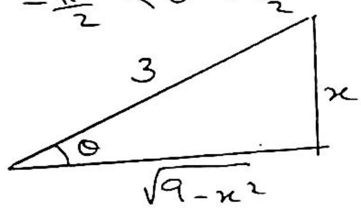


(The triangle <u>comes from $x = 2\tan\theta \Rightarrow \tan\theta = \frac{x}{2}$</u>).

<u>Example:</u> Evaluate $\int \frac{x^2\, dx}{\sqrt{9-x^2}}$.

$\cos\theta > 0$

1) Let $x = 3\sin\theta$, $dx = 3\cos\theta\, d\theta$, $\overbrace{-\frac{\pi}{2} < \theta < \frac{\pi}{2}}$



$x = 3\sin\theta$

Now $9 - x^2 = 9 - 9\sin^2\theta = 9\cos^2\theta$.

Then $\int \frac{x^2\, dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta \cdot 3\cos\theta\, d\theta}{|3\cos\theta|}$.

$= 9\int \sin^2\theta\, d\theta = 9\int \frac{1-\cos 2\theta}{2}\, d\theta = \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C$

$= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C = \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3} \cdot \frac{\sqrt{9-x^2}}{3}\right) + C$

Example:

Evaluate $\displaystyle\int \frac{dx}{\sqrt{25x^2-4}}$, $\quad x > \frac{2}{5}$.

$$\sqrt{25x^2-4} = \sqrt{25\left(x^2-\frac{4}{25}\right)} = 5\sqrt{x^2-\left(\frac{2}{5}\right)^2}.$$

Now, Let $\quad x = \frac{2}{5}\sec\theta, \quad dx = \frac{2}{5}\sec\theta\tan\theta\, d\theta$

on $\quad 0 < \theta < \frac{\pi}{2}$ $\qquad \frac{x}{a} \geq 1 \Rightarrow \frac{x}{\frac{2}{5}} \geq 1$

$\qquad \Rightarrow x \geq \frac{2}{5}$

$$x^2 - \left(\frac{2}{5}\right)^2 = \frac{4}{25}\sec^2\theta - \frac{4}{25} = \frac{4}{25}\tan^2\theta.$$

$\tan\theta > 0$ on $0<\theta<\frac{\pi}{2}$

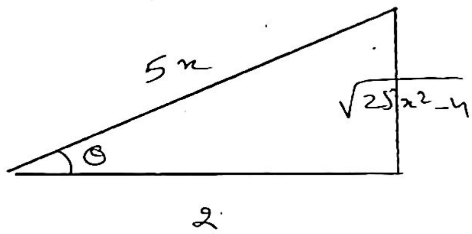
$$\sqrt{x^2-\left(\frac{2}{5}\right)^2} = \frac{2}{5}|\tan\theta| = \frac{2}{5}\tan\theta$$

Now: $\displaystyle\int \frac{dx}{\sqrt{25x^2-4}} = \int \frac{dx}{5\sqrt{x^2-\left(\frac{4}{25}\right)}}$

$$= \int \frac{\frac{2}{5}\sec\theta\tan\theta\, d\theta}{5\cdot\left(\frac{2}{5}\right)\tan\theta} = \frac{1}{5}\int \sec\theta\, d\theta$$

$$= \frac{1}{5}\ln|\sec\theta + \tan\theta| + C_1$$

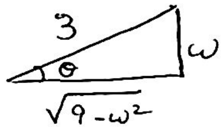
$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2-4}}{2}\right| + C_1.$$



(8.3) (20) Evaluate $\int \frac{\sqrt{9-w^2}}{w^2}\,dw$

Let $w = 3\sin\theta \quad , \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$dw = 3\cos\theta\, d\theta$



then $\sqrt{9-w^2} = 3\cos\theta$.

$$\Rightarrow \int \frac{\sqrt{9-w^2}}{w^2}\,dw = \int \frac{3\cos\theta \cdot 3\cos\theta\, d\theta}{9\sin^2\theta} = \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \int \frac{(1-\sin^2\theta)}{\sin^2\theta}\,d\theta = \int (\csc^2\theta - 1)\,d\theta = -\cot\theta - \theta + C$$

$$= -\frac{\cos\theta}{\sin\theta} - \theta + C = -\frac{\sqrt{9-w^2}}{w} - \sin^{-1}\left(\frac{w}{3}\right) + C$$

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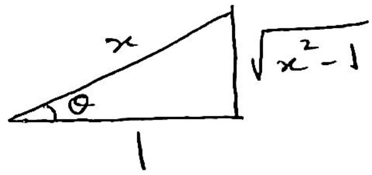
(26) $\int \frac{x^2\,dx}{(x^2-1)^{5/2}} \quad , \quad x > 1$



Let $x = \sec\theta \quad , \quad 0 < \theta < \frac{\pi}{2}$

$dx = \sec\theta\tan\theta\, d\theta$

$(x^2-1)^{5/2} = \tan^5\theta$.

$$\int \frac{x^2}{(x^2-1)^{5/2}}\,dx = \int \frac{\sec^3\theta \overset{\tan\theta}{\nearrow}\, d\theta}{\tan^5\theta} = \int \frac{\cos\theta}{\sin^4\theta}\,d\theta = \cdots$$

Let $u = \sin\theta \Rightarrow du = \cos\theta\, d\theta$

$$\cdots = \int \frac{\cos\theta\, d\theta}{\sin^4\theta} = \int \frac{du}{u^4} = -\frac{1}{3u^3} + C = \frac{-1}{3\sin^3\theta} + C$$

$$= \frac{-x^3}{3(x^2-1)^{3/2}} + C$$

(46) $\int \sqrt{\frac{x}{1-x^3}}\,dx$ (Hint: Let $u = x^{3/2}$
$du = \frac{3}{2}x^{\frac{1}{2}}dx$
$\Rightarrow dx = \frac{2}{3}u^{-\frac{1}{3}}du$)

$$\int \sqrt{\frac{u^{2/3}}{1-u^2}} \cdot \frac{2}{3}u^{-\frac{1}{3}}du$$

$$= \frac{2}{3}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{2}{3}\sin^{-1}u + C_1$$

$$= \frac{2}{3}\sin^{-1}(x^{3/2}) + C_1.$$

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(54) Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y = \pm b\sqrt{1-\frac{x^2}{a^2}}$$

$$A = 4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx = 4b\int_0^a \sqrt{1-\frac{x^2}{a^2}}\,dx$$

$x = a\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$dx = a\cos\theta\,d\theta$, $\sqrt{1-\frac{x^2}{a^2}} = \cos\theta$

$x = 0 = a\sin\theta \Rightarrow \theta = 0$,

$x = a = a\sin\theta \Rightarrow \theta = \frac{\pi}{2}$.

$$\Rightarrow 4b\int_0^a \sqrt{1-\frac{x^2}{a^2}}\,dx = 4b\int_0^{\pi/2}\cos\theta\,(a\cos\theta)\,d\theta$$

$$= 4ab\int_0^{\pi/2}\cos^2\theta\,d\theta = 4ab\int_0^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta = \boxed{\pi ab}$$

8.4 Integration of Rational Functions by Partial Fractions:

Example: Evaluate $\displaystyle\int \frac{5x-3}{(x+1)(x-3)}\,dx.$

$x^2 - 2x - 3$

we will Use the method of partial fractions:

$$\frac{5x-3}{x^2-2x-3} = \frac{5x-3}{(x+1)(x-3)} = \frac{A}{(x+1)} + \frac{B}{(x-3)}$$

$$= \frac{A(x-3) + B(x+1)}{(x+1)(x-3)} = \frac{(A+B)x - 3A+B}{(x+1)(x-3)}$$

$$\Rightarrow \quad A+B = 5 \quad \Rightarrow \quad \left.\begin{aligned} -A-B &= -5 \\ -3A+B &= -3 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} A &= 2 \\ B &= 3 \end{aligned}$$

$\&\quad -3A + B = -3$

$$\therefore \quad \frac{5x-3}{x^2-2x-3} = \frac{2}{(x+1)} + \frac{3}{(x-3)}$$

$$\Rightarrow \int \frac{5x-3}{x^2-2x-3}\,dx = \int \frac{2}{(x+1)}\,dx + \int \frac{3}{(x-3)}\,dx$$

$$= 2\ln|x+1| + 3\ln|x-3| + C.$$

Note: for $\frac{f(x)}{g(x)}$, the degree of $f(x)$ should be Less than the degree of $g(x)$.. If not divide & take Reminder term.

Method of Partial Fractions $\left(\frac{f(x)}{g(x)}\right)$ (proper)

1) Let $(x-r)$ be a linear factor of $g(x)$.

Suppose that $(x-r)^m$ is the heighst power of $(x-r)$ that divides $g(x)$. Then, to this factor: assign

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m} = \frac{\overset{\text{degree} < m}{*}}{(x-r)^m}.$$

2) Let x^2+px+q has irreducible quadratic factor of $g(x)$.

so that x^2+px+q has no roots.

Suppose $(x^2+px+q)^n$ is the heighst power of this factor that divides $g(x)$, then assign:

$$\frac{B_1x+C_1}{x^2+px+q} + \frac{B_2x+C_2}{(x^2+px+q)^2} + \cdots + \frac{B_nx+C_n}{(x^2+px+q)^n}.$$

3) Set the original fraction $\frac{f(x)}{g(x)}$ equal to the sum of all these partial fractions

4) Equate the coefficients of corresponding powers of x and solve the resulting equations.

Example: Use partial fraction to Evaluate:

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx.$$

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}.$$

$$= \frac{A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)}{(x-1)(x+1)(x+3)}$$

$$\Rightarrow \quad x^2+4x+1 = A(x^2+4x+3) + B(x^2+2x-3) + C(x^2-1)$$

$$= (A+B+C)x^2 + (4A+2B)x + (3A-3B-C)$$

$$\Rightarrow \left.\begin{array}{l} A+B+C = 1 \\ 4A+2B = 4 \\ 3A-3B-C = 1 \end{array}\right\} \Rightarrow A = \frac{3}{4},\ B = \frac{1}{2},\ C = \frac{-1}{4}.$$

$$\Rightarrow \int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx = \int \left[\frac{3}{4}\cdot\frac{1}{x-1} + \frac{1}{2}\cdot\frac{1}{x+1} - \frac{1}{4}\cdot\frac{1}{x+3}\right]dx$$

$$= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + K.$$

Example: Use partial fractions to Evaluate

$$\int \frac{6x+7}{(x+2)^2}\,dx$$

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$$6x+7 = A(x+2)+B = Ax + (2A+B)$$

$\Rightarrow A = 6$ & $B = -5$.

$$\therefore \int \frac{6x+7}{(x+2)^2}\,dx = 6\int \frac{dx}{x+2} - 5\int \frac{1}{(x+2)^2}\,dx$$

$$= 6\ln|x+2| + 5(x+2)^{-1} + C_1$$

Example: Use partial fraction to evaluate $\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx$

$$\Rightarrow \int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx =$$

$$\begin{array}{r} 2x \\ x^2-2x-3 \;\big)\; 2x^3-4x^2-x-3 \\ 2x^3-4x^2-6x \\ \hline 5x-3 \end{array}$$

$$= \int 2x\,dx + \int \frac{5x-3}{(x^2-2x-3)}\,dx$$

$$= \int 2x\,dx + \int \frac{5x-3}{(x+1)(x-3)}\,dx = \int\left(\frac{2}{x+1}\right)+\left(\frac{3}{x-3}\right)dx$$

we did it before.

$$= x^2 + 2\ln|x+1| + 3\ln|x-3| + C_1$$

Example: Evaluate $\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$.

Note that the denominator has an irreducible quadratic form.

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}.$$

$$\Rightarrow -2x+4 = (Ax+B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1)$$

$$= (A+C)x^3 + (-2A+B-C+D)x^2 + (A-2B+C)x + (B-C+D).$$

$$\Rightarrow \left.\begin{aligned} A+C &= 0 \\ -2A+B-C+D &= 0 \\ A-2B+C &= -2 \\ B-C+D &= 4 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= 2 \\ C &= -2 \\ B &= 1 \\ D &= 1 \end{aligned}$$

$$\Rightarrow \int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx = \int\left(\frac{2x+1}{x^2+1} + \frac{-2}{x-1} + \frac{1}{(x-1)^2}\right)dx$$

$$= \int \frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\,dx$$

$$= \ln(x^2+1) + \tan^{-1}x - 2\ln|x-1| - \frac{1}{(x-1)} + C$$

Example:

$$\int \frac{dx}{x(x^2+1)^2}$$

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$\frac{1}{x(x^2+1)^2} = \frac{A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x}{x(x^2+1)^2}$$

$\Rightarrow A = 1, \quad B = -1, \quad C = 0, \quad D = -1, \quad E = 0$

$$\Rightarrow \int \frac{dx}{x(x^2+1)^2} = \int \left(\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2} \right) dx.$$

$$= \ln|x| + \frac{-1}{2}\int \frac{2x}{x^2+1} - \frac{1}{2}\int \frac{2x}{(x^2+1)^2}\,dx$$

Let $u = x^2+1 \Rightarrow du = 2x\,dx$. . .

$$= \ln|x| - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2}$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + K.$$

$$= \ln \frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + K.$$

The Heaviside "Cover up" Method for linear factors:

Example: Find A, B and C in the following:

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

sol: Multiply both sides by $(x-1)$:

$$\frac{x^2+1}{(x-2)(x-3)} = A + \frac{B(x-1)}{(x-2)} + \frac{C(x-1)}{(x-3)}$$

- set $\boxed{x=1}$, we have:

$$\frac{(1)^2+1}{(1-2)(1-3)} = A \Rightarrow \boxed{A=1}$$

Similarly: Multiply both sides by $(x-2)$:

$$\frac{x^2+1}{(x-1)(x-3)} = \frac{A(x-2)}{(x-1)} + B + \frac{C(x-2)}{(x-3)}$$

- set $\boxed{x=2}$, we have:

$$\frac{(2)^2+1}{(2-1)(2-3)} = B \Rightarrow \boxed{B=-5}$$

Similarly: Multiply both sides by $(x-3)$:

- set $\boxed{x=3}$

$\Rightarrow \boxed{C=5}$

$$\therefore \frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3}.$$

Example:

$$\int \frac{x+4}{x^3+3x^2-10x}\,dx$$

$$\int \frac{x+4}{x(x^2+3x-10)}\,dx = \int \frac{x+4}{x(x-2)(x+5)}\,dx$$

$$\frac{x+4}{x(x-2)(x+5)} = \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{x+5}.$$

$$\frac{x+4}{(x-2)(x+5)} = A + \frac{Bx}{(x-2)} + \frac{Cx}{(x+5)}$$

- set $\boxed{x=0}$.

$$\boxed{\frac{4}{-10} = A}$$

$$\frac{x+4}{x(x+5)} = \frac{A(x-2)}{x} + B + \frac{C(x-2)}{x+5}$$

- set $\boxed{x=2}$, $\boxed{B = \frac{6}{14}}$

$$\frac{x+4}{x(x-2)} = \frac{A(x+5)}{x} + \frac{B(x+5)}{(x-2)} + C$$

- set $\boxed{x=-5}$ $\Rightarrow$ $\boxed{C = \frac{-1}{35}}$

$$\Rightarrow \int \frac{x+4}{x^3+3x^2-10x}\,dx = -\frac{2}{10}\ln|x| + \frac{3}{7}\ln|x-2| + \frac{-1}{35}\ln|x+5| + C.$$

Other Ways to determine the Coefficients:

Example:

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}.$$

by clearing ~~factors~~ fractions, differentiating the result & substituting $x=-1$.

Sol: 1 We first clear fractions:

$$(x-1) = A(x+1)^2 + B(x+1) + C.$$

substituting $\boxed{x=-1} \Rightarrow \boxed{-2 = C}$

2 Differentiate: $1 = 2A(x+1) + B.$

substitute $\boxed{x=-1} \Longrightarrow \boxed{B = 1}$

3 Differentiate: $0 = 2A \Longrightarrow \boxed{A = 0}$

$$\Rightarrow \frac{x-1}{(x+1)^3} = \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}.$$

Example: Find A, B and C in the expression:

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$$

$$x^2+1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

substitute $x = 1, 2, 3$ successively:

$2 = 2A \Longrightarrow \boxed{A=1}$

$5 = -B \Rightarrow \boxed{B=-5}$

$10 = 2C \Rightarrow \boxed{C=5}$

(8.4) (18) $\int_{-1}^{0} \frac{x^3\,dx}{x^2-2x+1}$

$$x^2-2x+1 \overline{\smash{\big)}\,x^3} \quad x+2$$

$$\begin{array}{l} x^3-2x^2+x \\ \hline 2x^2-x \\ 2x^2-4x+2 \\ \hline 3x-2 \end{array}$$

$$= \int_{-1}^{0} (x+2)\,dx + \int_{-1}^{0} \frac{3x-2}{(x-1)^2}\,dx$$

Now: $\frac{3x-2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$

$\Rightarrow \quad 3x-2 = A(x-1)+B$

$\Rightarrow \quad Ax + (-A+B) = 3x-2 \Rightarrow \boxed{A=3}$ & $\boxed{B=1}$

Then $\int_{-1}^{0} (x+2)\,dx + \int_{-1}^{0} \frac{3}{(x-1)} + \frac{1}{(x-1)^2}\,dx$

$$= \left.\frac{x^2}{2}+2x\right]_{-1}^{0} + \left.3\ln|x-1|\right|_{-1}^{0} + \left.\frac{-1}{x-1}\right]_{-1}^{0} = 2-3\ln 2$$

(30) $\int \frac{x^2+x}{x^4-3x^2-4}\,dx = \int \frac{A}{(x-2)} + \frac{B}{(x+2)} + \frac{Cx+D}{(x^2+1)}\,dx$

solve for A, B, C, D.

$x^2+x = A(x+2)(x^2+1) + B(x-2)(x^2+1) + (Cx+D)(x^2-4)$

$\Rightarrow x^2+x = (A+B+C)x^3 + (2A-2B+D)x^2 + (A+B-4C)x$

$+ 2A-2B-4D$

Then $A = \frac{3}{10}$, $B = \frac{-1}{10}$, $C = \frac{-1}{5}$, $D = \frac{1}{5}$

$\Rightarrow \int \frac{x^2+x}{x^4-3x^2-4}\,dx = \frac{3}{10}\ln|x-2| - \frac{1}{10}\ln|x+2| + \frac{-1}{10}\ln|x^2+1| + \frac{1}{5}\tan^{-1}x$

$+ C$

(47) $\displaystyle\int \frac{\sqrt{x+1}}{x}\,dx$, $\left(\text{Hint : let } x+1 = u^2\right)$

$dx = 2u\,du$

$$= \int \frac{\sqrt{u^2}}{u^2-1} \cdot 2u\,du = \int \frac{2u^2}{u^2-1}\,du =$$

$$\begin{array}{r|l} & \ \ 2 \\ u^2-1 & \overline{2u^2} \\ & \underline{2u^2-2} \\ & \ \ 2 \end{array}$$

$$= \int 2\,du + \int \frac{2}{(u-1)(u+1)}\,du \quad \text{--- } (*)$$

Now: $\dfrac{2}{(u-1)(u+1)} = \dfrac{A}{u+1} + \dfrac{B}{u-1}$

$2 = A(u-1) + B(u+1) = (A+B)u + (-A+B)$

$\Rightarrow A+B = 0 \quad \& \quad -A+B = 2$

$\Rightarrow \boxed{A = -1} \quad \& \quad \boxed{B = 1}$

$$\Rightarrow (*) = 2u + \int \frac{-1}{u+1} + \frac{1}{u-1}\,du$$

$$= 2u - \ln|u+1| + \ln|u-1| + C$$

$$= 2\sqrt{(x+1)} - \ln|\sqrt{(x+1)}+1| + \ln|\sqrt{x+1}-1| + C$$