Examples:

① $f(x) = x^2 + x^4 + 4$

$f'(x) = 2x + 4x^3$.

---

② $f(x) = \dfrac{x+1}{x^2-1}$

$f'(x) = \dfrac{(x^2-1)(1) - (x+1)(2x)}{(x^2-1)^2} = \dfrac{x^2-1-2x^2-2x}{(x^2-1)^2}$

$= \dfrac{-x^2-2x-1}{(x^2-1)^2}$.

---

③ $\dfrac{d}{dx}(x^3+2x)^4 = 4(x^3+2x)^3 \cdot (3x^2+2)$.

---

Example: Find the Horizontal tangent for $f(x) = x^4 - 2x^2 + 2$?

Horizontal tangent occurs at $f'(x) = 0$.

$\Rightarrow f'(x) = 4x^3 - 4x = 4x(x^2-1) = 0$

$\Rightarrow f'(x) = 0$ iff $x = 0, 1, -1$.

---

## 3.3 Derivatives of Trigonometric functions:

1) $(\sin x)' = \cos x$

2) $(\cos x)' = -\sin x$

3) $(\tan x)' = \sec^2 x.$

4) $(\sec x)' = \sec x \tan x$

5) $(\csc x)' = -\csc x \cot x$

6) $(\cot x)' = -\csc^2 x.$

Examples: ① $y = x^2 \cdot \sin x$

$y' = x^2 \cos x + 2x \sin x$

② $y = \tan(x^3)$

$y' = \sec^2(x^3) \cdot (3x^2) = 3x^2 \sec^2(x^3).$

③ $\frac{d}{dx}(\cos(\sqrt{x})) = -\sin(\sqrt{x}) \cdot (\sqrt{x})' = -\sin(\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right)$

$= -\frac{1}{2\sqrt{x}} \sin(\sqrt{x})$

④ $\frac{d}{dx}(\sec x \tan x)$

$= \sec x\,(\tan x)' + \tan x\,(\sec x)'$

$= \sec x \cdot \sec^2 x + \tan x \sec x \tan x$

$= \sec^3 x + \sec x \tan^2 x.$

---

⑤ $y = \sqrt{f(x)}$ , $f(x) > 0$

$y' = \left[(f(x))^{\frac{1}{2}}\right]' = \frac{1}{2}(f(x))^{\frac{1}{2}-1}(f'(x))$

$= \frac{1}{2} \cdot \frac{1}{\sqrt{f(x)}} \cdot f'(x).$

---

⑥ $f(x) = \sqrt{x^2+1} = (x^2+1)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(x^2+1)^{\frac{1}{2}-1} \cdot (2x) = \frac{x}{\sqrt{x^2+1}}$

---

⑦ $\frac{d}{dt}\left(\frac{\tan t}{1+\sec t}\right) = \frac{(1+\sec t)(\sec^2 t) - (\tan t)(\sec t \tan t)}{(1+\sec t)^2}$

---

Example: Find the equation of the tangent line to the curve $f(x) = \sec x \tan x$ at $x = \frac{\pi}{4}$.

Then Find the Normal Line to $f(x)$ at $x = \frac{\pi}{4}$.

sol: Equation of the Line is

$$y - y_1 = m(x - x_1) \quad , \quad m = \text{slope}.$$

$$x_1 = \frac{\pi}{4} \Rightarrow y_1 = f(x_1) = f\left(\frac{\pi}{4}\right) = \sec\frac{\pi}{4} \tan\frac{\pi}{4} = (\sqrt{2})(1) = \sqrt{2}.$$

$$m = f'\left(\frac{\pi}{4}\right) \Rightarrow f'(x) = \sec^3 x + \tan^2 x \sec x$$

$$\Rightarrow f'\left(\frac{\pi}{4}\right) = \sec^3\frac{\pi}{4} + \tan^2\frac{\pi}{4} \sec\frac{\pi}{4}$$

$$= (\sqrt{2})^3 + (1)^2(\sqrt{2})$$

$$= 2\sqrt{2} + \sqrt{2} = \boxed{3\sqrt{2}}$$

$\Rightarrow$ Tangent Line: $y - \sqrt{2} = 3\sqrt{2}\left(x - \frac{\pi}{4}\right)$

$$\Rightarrow y = 3\sqrt{2}\, x - 3\sqrt{2}\left(\frac{\pi}{4}\right) + \sqrt{2}$$

Normal Line: $y - y_1 = \frac{-1}{m}(x - x_1)$

$$y - \sqrt{2} = \frac{-1}{3\sqrt{2}}\left(x - \frac{\pi}{4}\right)$$

## 3.4 Implicit differentiation:

1) Differentiate both sides of the equation with respect to $x$, treating $y$ as a differentiable function of $x$.

2) Collect the terms $\frac{dy}{dx}$ on one side of the equation and solve $\frac{dy}{dx}$.

Example: Find $\frac{dy}{dx}$ if $y^2 = x^2 + \sin xy$.

$$2y\,y' = 2x + \cos(xy)\left[x\frac{dy}{dx} + y\right]$$

$$2y\,y' = 2x + x\cos(xy)\frac{dy}{dx} + y\cos(xy)$$

$$\Rightarrow \quad 2yy' - x\cos(xy)\,y' = 2x + y\cos(xy)$$

$$y'(2y - x\cos(xy)) = 2x + y\cos(xy)$$

$$\Rightarrow \quad y' = \frac{2x + y\cos(xy)}{2y - x\cos(xy)}.$$

Example: Find $d^2y/dx^2$ if $2x^3 - 3y^2 = 8$.

$$6x^2 - 6yy' = 0.$$

$$\Rightarrow \quad 6yy' = 6x^2$$

$$\Rightarrow \quad y' = \frac{x^2}{y}$$

Now: $$y'' = \frac{y(2x) - (x^2)y'}{y^2} = \frac{2xy - x^2\left(\frac{x^2}{y}\right)}{y^2}$$

$$= \frac{2x\frac{y^2}{y} - \frac{x^4}{y}}{y^2} = \frac{2xy^2 - x^4}{y^3}.$$

Example: Find $y'$ if $xy = \cot(xy)$.

$$x \cdot y' + 1 \cdot y = -\csc^2(xy)(xy' + y).$$

$$xy' + y = -xy'(\csc^2(xy)) - y\csc^2(xy)$$

$$\Rightarrow \quad (x + x\csc^2(xy))y' = -y\csc^2(xy) - y$$

$$\Rightarrow \quad y' = \frac{-y(1+\csc^2(xy))}{x(1+\csc^2(xy))} = \frac{-y}{x}.$$

3.5 Linearization and Differentials:

Def: If $f$ is differentiable at $x = a$,

then the approximation function

$$L(x) = f(a) + f'(a)(x-a)$$

is the linearization of $f$ at $a$.

The approximation $f(x) \approx L(x)$

of $f$ by $L$ is the standard Linear approximation

of $f$ at $a$. $a$ is the center of the approximation.

Example: Find the Linearization of $f(x) = \sqrt{1+x}$

at $x = 0$.

$L(x) = f(0) + f'(0)(x-0)$.

$f(0) = \sqrt{1+0} = \sqrt{1} = 1$

$f'(x) = \dfrac{1}{2\sqrt{1+x}} \Rightarrow f'(0) = \dfrac{1}{2\sqrt{1+0}} = \dfrac{1}{2(1)} = \dfrac{1}{2}$

$\Rightarrow L(x) = 1 + \frac{1}{2}x \approx f(x) = \sqrt{1+x}$

When $x = 0.002 \Rightarrow \sqrt{1+0.002} \approx 1 + \frac{1}{2}(0.002) = 1.001$.

**Example:** Find the Linearization of $f(x) = \sec x$ at $x = \frac{\pi}{4}$.

$$L(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)$$

$$f\left(\frac{\pi}{4}\right) = \sec\left(\frac{\pi}{4}\right) = \sqrt{2}.$$

$$f'(x) = \sec x \tan x \Longrightarrow f'\left(\frac{\pi}{4}\right) = \sec\frac{\pi}{4}\tan\frac{\pi}{4}$$
$$= \sqrt{2}(1) = \sqrt{2}.$$

$$\therefore L(x) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right)$$

We conclude: $\sec x \approx \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right)$ at $x = \frac{\pi}{4}$.

---

**Example:** Approximate $\sqrt[3]{8.01}$.

$$f(x) = \sqrt[3]{x} \quad \text{at } a = 8$$

$$f(a) = f(8) = \sqrt[3]{8} = 2$$

$$f'(x) = \frac{1}{3}x^{-2/3} \Rightarrow f'(8) = \frac{1}{3}(8)^{-2/3} = \frac{1}{12}.$$

$$\Rightarrow L(x) = f(8) + f'(8)(x-8)$$
$$= 2 + \frac{1}{12}(x-8) \approx \sqrt[3]{x}$$

$$\Rightarrow \sqrt[3]{8.01} \approx 2 + \frac{1}{12}(8.01 - 8) \cong 2.0008$$

## Differentials:

**Def:** Let $y = f(x)$ be differentiable function.

- The differential $dx$ is an Independent Variable
- The differential $dy$ is $\boxed{dy = f'(x)\,dx}$

**Note:** If the ratio of $dy$ and $dx$ exists, it is equal to the derivative.

$dy = f'(x)\,dx$ is differential $\boxed{\approx \Delta f}$

while $\dfrac{dy}{dx} = f'(x)$ is derivative.

**Example:** ⓐ Find $dy$ if $y = x^5 + 37x$.

ⓑ Find the value of $dy$ when $x=1$ and $dx=0.2$

sol:

ⓐ $dy = (5x^4 + 37)\,dx$.

ⓑ $dy = (5(1)^4 + 37)(0.2) = (42)(0.2) = 8.4$.

Example: The radius $r$ of a circle increases from $r = 10$ to $10.1$. Use $dA$ to estimate the increase in the circle's area. Estimate the area of the enlarged circle and compare your estimate to the true area found by direct calculation.

Sol: Area of circle:

$$A = \pi r^2.$$

$$\Delta A \approx dA = 2\pi r \underbrace{dr}_{\approx \Delta r} = 2\pi (10)(0.1) = 2\pi.$$

When $r = 10 \Rightarrow$ Area $= \pi (10)^2 = 100\pi$.

Exact Area for the enlarged $= \pi (10.1)^2 = 102.01\pi\ m^2$

Approximate area $\approx A + dA = 100\pi + 2\pi = 102\pi\ m^2$.

$\Rightarrow$ Error $= |$Exact $-$ Approximation$|$

$$= |102.01\pi - 102\pi| = 0.01\pi.$$

## Chapter 4 : Applications of derivatives:

### 4.1 Increasing and decreasing functions:

Def: Let $f(x)$ be a function defined on an Interval $I$.

(a) $f$ is <u>increasing</u> on $I$ whenever $x_2 > x_1$, then

$f(x_2) > f(x_1)$, for all $x_1, x_2 \in I$.

(b) $f$ is <u>decreasing</u> on $I$ whenever $x_2 > x_1$, then

$f(x_2) < f(x_1)$, for all $x_1, x_2 \in I$

---

Thm: Suppose that $f$ is Continuous on $[a,b]$ and differentiable on $(a,b)$, then:

(a) If $f'(x) > 0$, $\forall x \in (a,b)$, then $f$ is Increasing on $[a,b]$.

(b) If $f'(x) < 0$, $\forall x \in (a,b)$, then $f$ is decreasing on $[a,b]$.

---

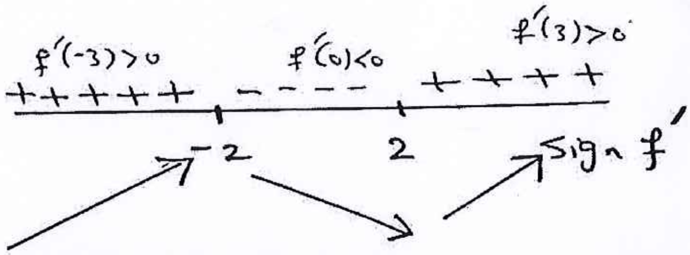
**Example:** Let $f(x) = x^3 - 12x - 5$.

Identify the Intervals on which $f$ is Increasing and decreasing.

**Sol:** $f'(x) = 3x^2 - 12 = 0$

$3(x^2 - 4) = 3(x-2)(x+2) = 0$

$\Rightarrow \quad x = 2, \; x = -2$




Then

$f$ is Increasing on $(-\infty, -2] \cup [2, \infty)$

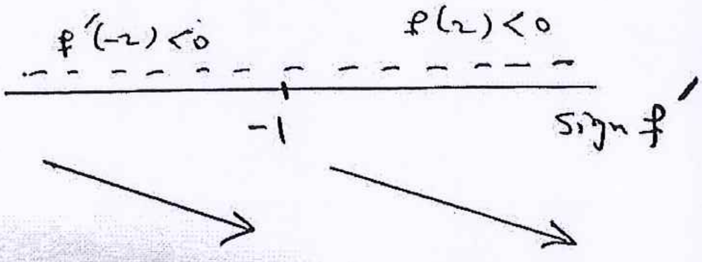
$f$ is decreasing on $[-2, 2]$

---

**Example:** Let $f(x) = 1 - (x+1)^3$. Identify the Intervals on which $f$ is Increasing and decreasing.

**Sol:** $f'(x) = 0 - 3(x+1)^2 = -3(x+1)^2 = 0$

$\Rightarrow \quad \boxed{x = -1}$




$\Rightarrow$ $f$ is decreasing on

$(-\infty, -1] \cup [-1, \infty) = (-\infty, \infty)$.

## 4.2 Extreme values of functions:

**Def**: Let $f$ be a function with domain $D$. Then:

قيمة عظمى مطلقة

(a) $f$ has an absolute maximum value on $D$ at a point $c$ if $f(x) \leq f(c)$, $\forall x \in D$.

قيمة صغرى مطلقة

(b) $f$ has an absolute minimum value on $D$ at a point $c$ if $f(x) \geq f(c)$, $\forall x \in D$.

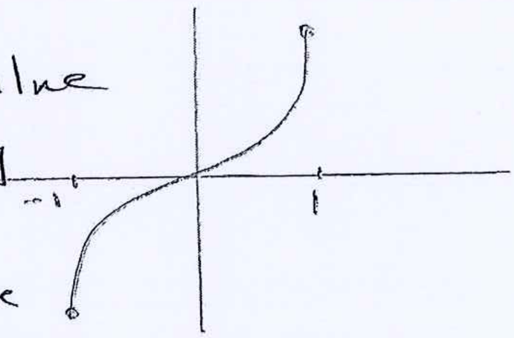
---

عظمى محلية  صغرى محلية

**Note**: $f(c)$ is called Local maximum (resp. local Min.) if the inequality in (a) (resp. (b)) holds in a small interval around $x = c$.

---

**Example**: $f(x) = x^3$, $D = [-1, 1]$




- $f(-1) = -1$ is absolute min value since $f(x) \geq f(-1)$, $\forall x \in [-1, 1]$
- $f(1) = 1$ is absolute Max value since $f(x) \leq f(1) = 1$, $\forall x \in [-1, 1]$.

**Note:** If $f(x) = x^3$, $D = (-1, 1)$, then $f$ has neither maximum nor minimum.

---

**Thm:** If $f$ is continuous function on a closed interval $[a, b]$ then $f$ has both an absolute maximum value and an absolute minimum value.

---

**Example:** $f(x) = x^3$, $D = [-1, 1]$.
$f$ has both absolute maximum $f(1) = 1$ and absolute minimum $f(-1) = -1$

---

**Note:** If we want to find the extreme values of a function $f$ on a closed interval, we look for these values at the endpoints of the Interval and at the interior points (نقاط داخلية) where $f' = 0$ or $f'$ is undefined. (critical points) (نقاط حرجة)

---

Def: An interior point of the domain of a function $f$ where $f' = 0$ or $f'$ is undefined is called critical point of $f$.

Example: Find the absolute Max. and Absolute Min. of $f(x) = x^{2/3}$ on $[-1, 8]$.

Sol: critical points: $f'(x) = \frac{2}{3} x^{\frac{2}{3}-1}$

$\Rightarrow f'(x) = \frac{2}{3x^{\frac{1}{3}}}$

critical point

$\Rightarrow f'(x) \neq 0$ & $f'(x)$ is undefined at $\boxed{x=0}$

$f(-1) = (-1)^{2/3} = \sqrt[3]{(-1)^2} = 1$

$f(0) = (0)^{2/3} = 0$ $\leftarrow$

$f(8) = (8)^{2/3} = \sqrt[3]{(8)^2} = \sqrt[3]{64} = 4$ $\leftarrow$

$\Rightarrow f(0) = 0$ is an absolute minimum value

$f(8) = 4$ is an absolute Maximum Value.

**Thm:** If $f$ is differentiable and has an extreme value at an interior point $c$, then $f'(c)=0$.

**Note:** If $f'(c)=0$, then this does not mean that $f$ has an extreme value (Max or Min) at $\boxed{x=c}$

**Example:** $f(x)=x^3$, $f'(x)=3x^2$

$f'(0)=3(0)^2=0$, but $0$ is neither Maximum Nor Minimum value of $f(x)$.

**Thm:** First derivative test: suppose that $f$ has a critical point at $x=c$ & $f'(x)$ exists in an open interval containing $x=c$. Then:

1) If $f'$ changes sign from positive ($+$) to negative ($-$) at $x=c$, then $f(c)$ is a local Maximum.

2) If $f'$ changes sign from negative ($-$) to positive ($+$) at $x=c$, then $f(c)$ is a local Minimum.

(c) If $f'$ does not change sign at $x=c$, then $f$ does not have an extreme value at $x=c$.

Example: Let $f(x) = x^4 - 2x^2$.

(a) Find the critical points of $f$.

(b) Identify the Intervals on which $f$ is Increasing and decreasing.

(c) Find the function's Local and absolute extreme Values.

Sol: Domain $(-\infty, \infty)$

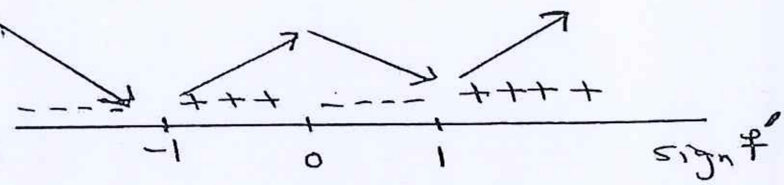
a) Critical points: $f'(x) = 4x^3 - 4x = 4x(x^2-1)$

$= 4x(x-1)(x+1) = 0$

then $x = 0$, $x = 1$, $x = -1$ are critical points of $f$.

$f$ is Increasing on $[-1, 0] \cup [1, \infty)$

$f$ is decreasing on $(-\infty, -1] \cup [0, 1]$




$f(-1) = -1$ is local Min. value $\longrightarrow$ (absolute Min)

$f(0) = 0$ is local Max. Value

$f(1) = -1$ is local Min. Value $\longrightarrow$ (absolute Min)

**Def:** ① If $f''(x) \geq 0$, $\forall x \in I$ then

مقعر للأعلى
$f$ is Concave up on $I$

If $f''(x) \leq 0$, $\forall x \in I$, then $f$ is مقعر للأسفل Concave down on $I$

---

**Def:** A point where $f$ has tangent Line and changes concavity is called an نقطة انعطاف Inflection point of $f$

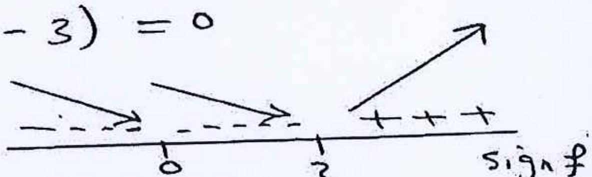
---

**Example:** Find the Intervals where the function $f(x) = x^4 - 4x^3 + 10$ is increasing, decreasing, concave up, concave down.

Sol: $D = (-\infty, \infty)$

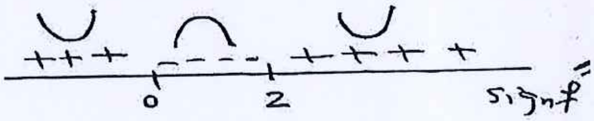
$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3) = 0$

$\Rightarrow$ critical points $x = 0, x = 3$




$f$ is Increasing on $[3, \infty)$ & decreasing on $(-\infty, 3]$

$f''(x) = 12x^2 - 24x = 12x(x-2) = 0$




$x = 2$, $x = 0$

$f$ is Concave up on $(-\infty, 0] \cup [2, \infty)$
$f$ is Concave down on $[0, 2]$
$\Rightarrow$ $(0, 10)$ & $(2, -6)$ are Inflection points.

**Theorem: Second derivative test**

Suppose that $f'(c) = 0$ and $f''$ is Continuous on an open Interval Containing $c$. Then:

(a) If $f''(c) < 0$, then $f(c)$ is a Local Maximum.

(b) If $f''(c) > 0$, then $f(c)$ is a Local Minimum.

(c) If $f''(c) = 0$, then the test fails.

**Example:** Use 2nd derivative test to find an extreme values of $f(x) = x^4 - 2x^2$.

**Sol:** $f'(x) = 4x^3 - 4x = 4(x)(x^2 - 1) = 0$

$\Rightarrow x = 0, \; x = 1, \; x = -1$

$f''(x) = 12x^2 - 4$

at $x = 0 \Rightarrow f''(0) = -4 < 0 \Rightarrow f(0) = 0$ is Local Max.

at $x = 1 \Rightarrow f''(1) = 8 > 0 \Rightarrow f(1) = -1$ is Local Min.

at $x = -1 \Rightarrow f''(-1) = 8 > 0 \Rightarrow f(-1) = -1$ is Local Min.

**Example:** Consider $f(x) = \dfrac{x^2}{x+1}$. sketch $f$.

$$f'(x) = \frac{x^2+2x}{(x+1)^2}, \quad f''(x) = \frac{2}{(x+1)^3}$$

1) Domain : $(-\infty, \infty)\setminus\{-1\}$

2) $\lim\limits_{x\to+\infty} \dfrac{x^2}{x+1} = +\infty = \lim \dfrac{x^2}{x(1+\frac{1}{x})}$

3) $\lim\limits_{x\to-\infty} \dfrac{x^2}{x+1} = -\infty$

4) Horizontal Asymptote : None.

5) $\lim\limits_{x\to-1^-} \dfrac{x^2}{x+1} = -\infty$

6) $\lim\limits_{x\to-1^+} \dfrac{x^2}{x+1} = +\infty$

$$\begin{array}{r|l} & x-1 \\ \hline x+1 & x^2 \\ & -x^2 \mp x \\ \hline & -x \\ & -x-1 \\ \hline & 1 \end{array}$$

7) Vertical Asymptote: $x=-1$

8) Oblique Asy. $y = x-1$.

9) Critical points : $f'(x)=0 \Rightarrow x = 0, -2.$ $\begin{cases} f'(x) \text{ DNE} \\ -1 \notin D \end{cases}$

10) $f$ is Increasing on $(-\infty,-2] \cup [0,\infty)$

$f$ is decreasing on $[-2,-1) \cup (-1,0]$

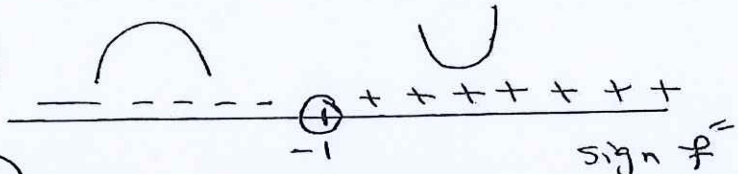
$$\begin{array}{ccccccc} +++ & & ---- & & ---- & & ++++ \\ & -2 & & -1 & & 0 & \text{sign } f' \\ \nearrow & & \searrow & & \searrow & & \nearrow \end{array}$$

11) $f(-2) = -4$ is local Max.

12) $f(0) = 0$ is Local Min.

13) $f$ is concave up on $(-1, \infty)$

& Concave down on $(-\infty, -1)$



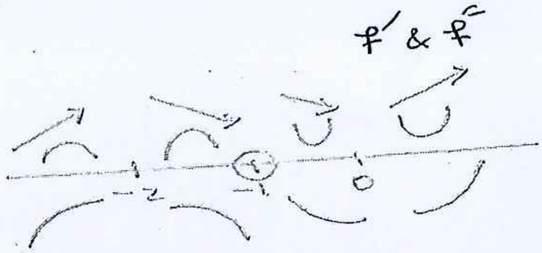

14) NO Absolute Max, NO Absolute Min.

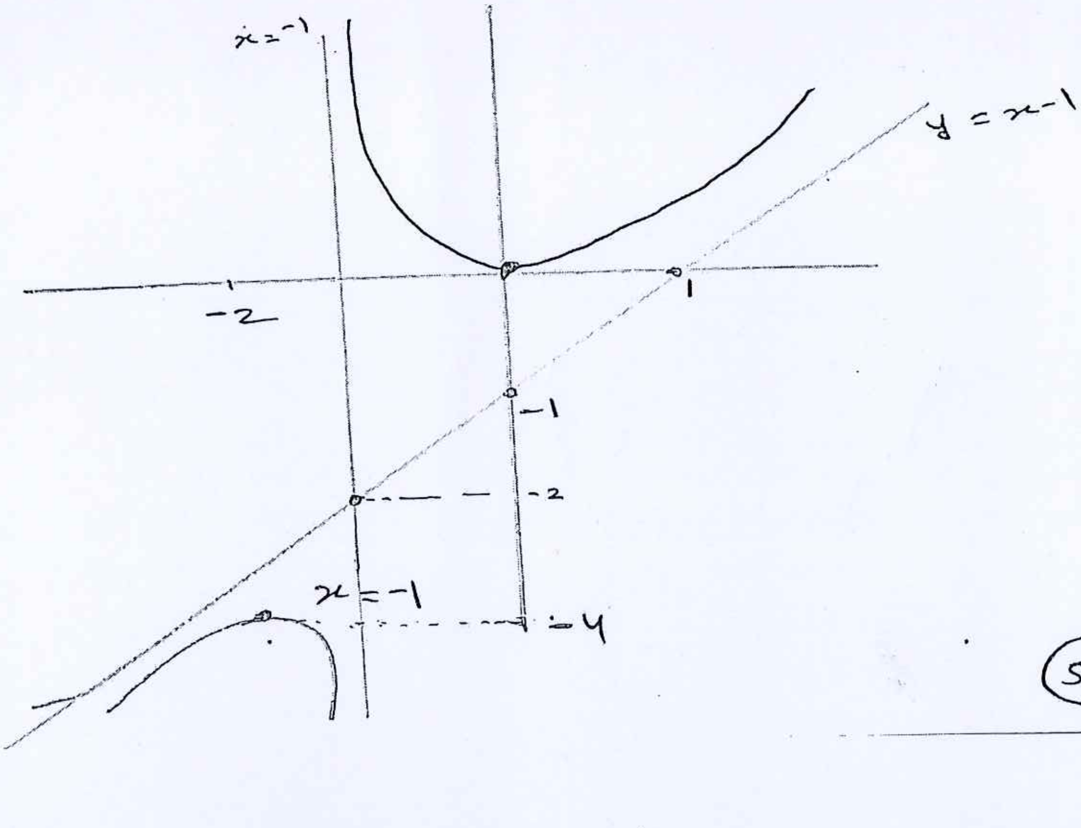
15) NO Inflection points.

16) Range: $(-\infty, -4] \cup [0, \infty)$

17) $x$-intercept: $y = 0 \Rightarrow x = 0$

18) $y$-intercept: $x = 0 \Rightarrow y = 0$



Example: $f(x) = \dfrac{x^2}{x^2-1}$ , sketch the graph $f$.

$$f'(x) = \frac{-2x}{(x^2-1)^2} \quad , \quad f''(x) = \frac{6x^2+2}{(x^2-1)^3} .$$

1) Domain : $(-\infty, \infty) \setminus \{\pm 1\}$

2) $\lim\limits_{x \to \pm\infty} \dfrac{x^2}{x^2-1} = 1$

3) H.A : $y = 1$

4) $\lim\limits_{x \to 1^-} \dfrac{x^2}{x^2-1} = -\infty$

5) $\lim\limits_{x \to 1^+} \dfrac{x^2}{x^2-1} = +\infty$

6) $\lim\limits_{x \to -1^-} \dfrac{x^2}{x^2-1} = +\infty$

7) $\lim\limits_{x \to -1^+} \dfrac{x^2}{x^2-1} = -\infty$

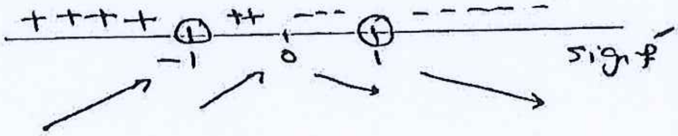
8) V.A : $x = 1$ & $x = -1$

9) critical points $f'(x) = 0 \Rightarrow x = 0$

$f'(x)$ is Undefined at $x = -1, 1 \notin D$.

10) $f$ is Increasing on $(-\infty,-1)\cup(-1,0]$

$f$ is decreasing on $[0,1)\cup(1,\infty)$




11) $f(0)=0$ is local Max.

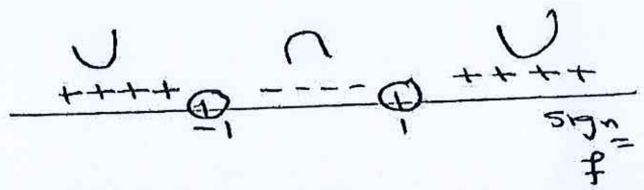
12) NO Local Min..

13) NO Absolute Max, NO Absolute Min.

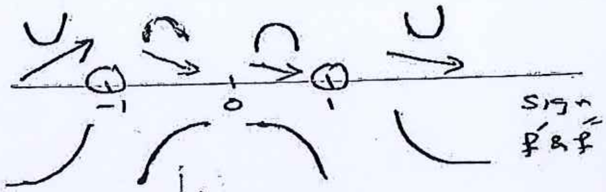
14) $f$ is concave up on

$(-\infty,-1)\cup(1,\infty)$

and concave down on $(-1,1)$.



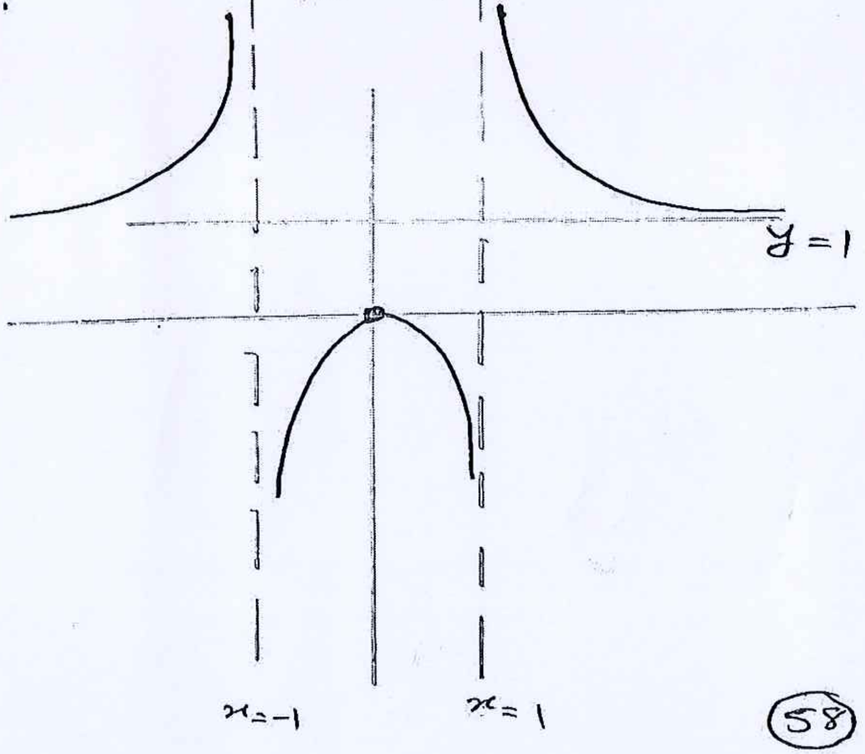

15) NO Inflection point




6) Range: $(-\infty,0]\cup(1,\infty)$.

7) x-intercept: $y=0$, $x=0$

8) y-intercept: $x=0$, $y=0$



Example: $f(x) = \dfrac{x}{x^2+1}$

$f'(x) = \dfrac{1-x^2}{(x^2+1)^2}$ , $f''(x) = \dfrac{2x(x^2-3)}{(x^2+1)^3}$.

1) Domain : $(-\infty, \infty)$

2) $\displaystyle\lim_{x \to \pm\infty} \frac{x}{x^2+1} = 0$

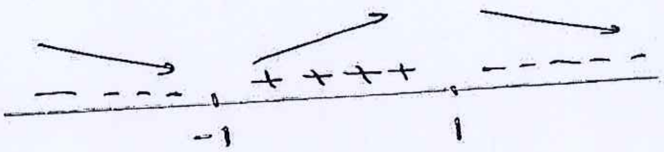
3) Horiz. Asym. $y = 0$

4) V.A: None

5) Oblique Asy. None.

6) Critical points : $x = 1, x = -1$

7) $f$ is Increasing on $[-1, 1]$




& decreasing on $(-\infty, -1] \cup [1, \infty)$

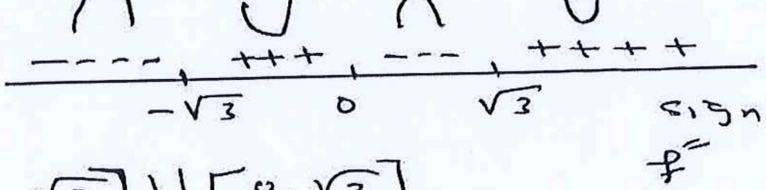
8) Local Max : $f(1) = \frac{1}{2}$

9) Local Min : $f(-1) = -\frac{1}{2}$

10) Absolute Max : $f(1) = \frac{1}{2}$

11) Absolute Min : $f(-1) = -\frac{1}{2}$

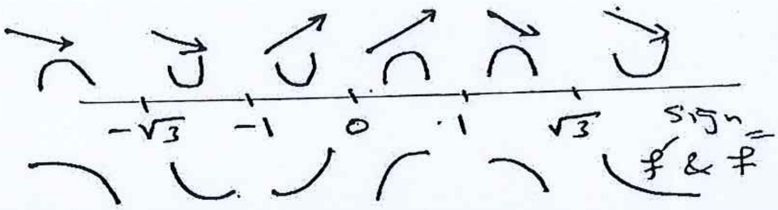
12) $f$ is concave up on $[-\sqrt{3},0] \cup [\sqrt{3},\infty)$

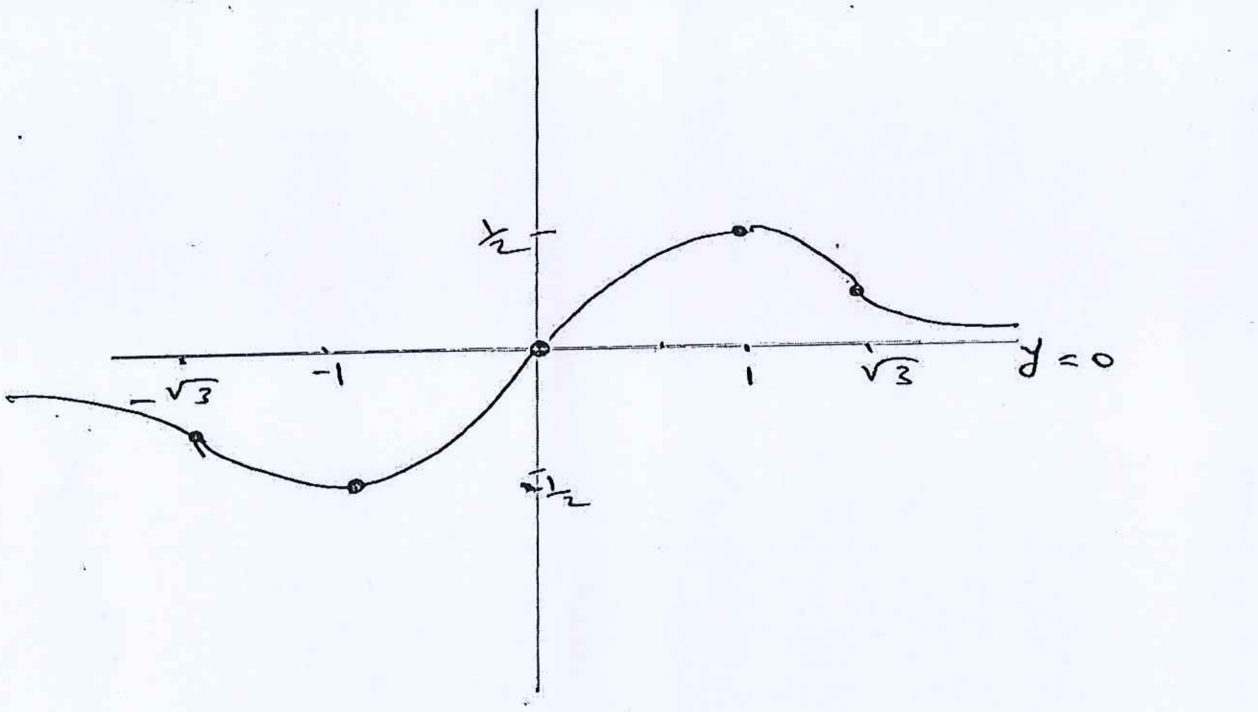



and concave down $(-\infty,-\sqrt{3}] \cup [0,\sqrt{3}]$

13) Inflection points $\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right), (0,0), \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$

14) Range. $[-\frac{1}{2}, \frac{1}{2}]$

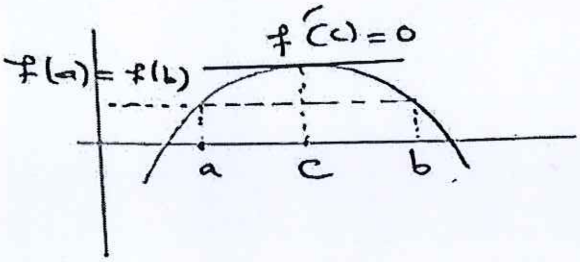


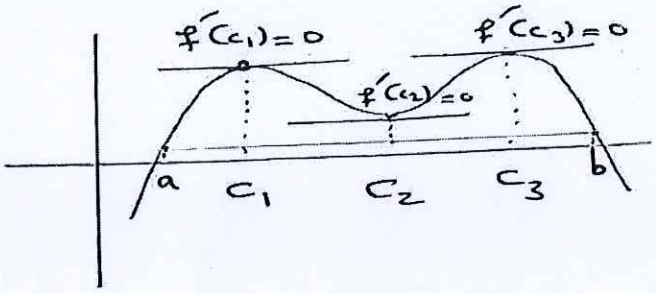



## 4.3 The Mean Value Theorem

**Theorem:** Rolle's Theorem:

If ① $f(x)$ is continuous on $[a, b]$ and ② diff. on $(a, b)$ and ③ $f(a) = f(b)$, then there is at least one point $c \in (a, b)$ such that $\underbrace{f'(c)}_{\text{slope}} = 0$.



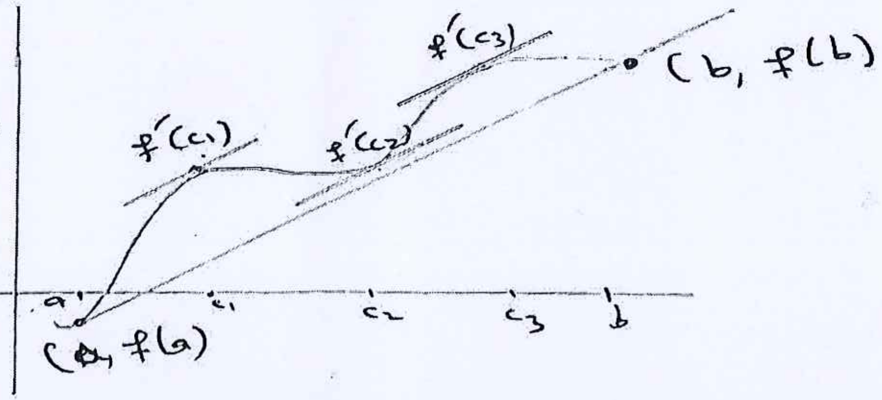

نظرية القيم المتوسطة

**Theorem:** Mean Value Theorem: (MVT)

If $f$ is continuous on $[a, b]$ & Diff. on $(a, b)$, Then: there is at least one $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

$$\text{slope} = \frac{f(b) - f(a)}{b - a}$$



Example: Let $f(x) = x^2$, $x \in [1, 4]$

Find the value(s) of $C$ in the Conclusion of the Mean Value Theorem.

Sol: 1) $f$ is Cont. on $[1,4]$ since it's polynomial.

2) $f$ is diff on $(1,4)$, since $f'(x) = 2x$.

$\Rightarrow$ by MVT, there is $c \in (1,4)$

such that:

$$f'(c) = \frac{f(4) - f(1)}{4-1}$$

$$\Rightarrow \quad 2c = \frac{16-1}{3} = \frac{15}{3}$$

$$\therefore \quad c = \frac{15}{6} = \frac{5}{2}.$$

Example: suppose $f'(x) \leq 1$, $\forall\ 1 \leq x \leq 4$.

show that $f(4) - f(1) \leq 3$.

Sol: $f$ is Cont. on $[1,4]$ & diff. on $(1,4) \Rightarrow \exists c \in (1,4)$

(s.t) $f'(c) = \dfrac{f(4) - f(1)}{4-1} \leq 1 \Rightarrow f(4) - f(1) \leq 3$.

# Chapter 5: Integration التكامل

## 5.1 Antiderivative and Integration: الاقتران البدائي

**Def:** A function $F$ is called an antiderivative of a function $f$ on an interval $I$ if

$$F'(x) = f(x) \quad , \forall x \in I \quad , \quad \left[\text{(i.e)}: \int f(x)\,dx = F(x)+C\right]$$

The set of all antiderivatives of $f$ is called the indefinite integral of $f$ & is denoted by $\int f(x)\,dx$.
تكامل غير محدود

**Example:** $F(x) = x^3$ is an antiderivative of $f(x) = 3x^2$

since $F'(x) = 3x^2 = f(x)$ $\quad$ $\left[\text{or}: \int 3x^2\,dx = x^3 + C\right]$

**Example:** (a) $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad , n \neq -1$

(b) $\displaystyle\int \sin x\,dx = -\cos x + C$

(c) $\displaystyle\int \cos x\,dx = \sin x + C$

$\displaystyle\int \sin(kx)\,dx = -\frac{\cos(kx)}{k} + C \quad , k \neq 0$

(d) $\int \sec^2 x \, dx = \tan x + C$

$\int \sec^2(kx)\, dx = \frac{\tan(kx)}{k} + C$ , $k \neq 0$.

(e) $\int \sec x \tan x \, dx = \sec x + C$

(f) $\int \csc x \cot x \, dx = -\csc x + C$

(g) $\int \csc^2 x \, dx = -\cot x + C$

Example: 1) $\int (x^{-2} - x^2 + 1)\, dx = \int x^{-2} dx - \int x^2 dx + \int 1 \, dx$

$= \frac{x^{-1}}{-1} - \frac{x^3}{3} + x + C = -\frac{1}{x} - \frac{x^3}{3} + x + C.$

2) $\int \cos^2\theta \, d\theta = \int \left( \frac{1+\cos 2\theta}{2} \right) d\theta$

$= \frac{1}{2} \int (1 + \cos 2\theta)\, d\theta = \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2} + C\right]$

$= \frac{1}{2}\theta + \frac{\sin 2\theta}{4} + C$

3) $\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx$

$= \int \sec^2 x \, dx - \int 1 \, dx = \tan x - x + C.$

التكامل المحدود — المساحات

## 5.2 Definite Integrals and Areas:

We denote for the Definite Integral by:

الحد العلوي للتكامل ← $b$
Upper Limit of Integration

$$\int_a^b f(x) \, dx$$

الحد السفلي للتكامل
Lower Limit of Integration

النظرية الأساسية في التفاضل والتكامل

**Theorem: Fundamental Theorem of Calculus.**

(I) Suppose that $f$ is Continuous on $[a,b]$ and $F$ is an antiderivative of $f$ on $[a,b]$ then:

$$\int_a^b f(x) dx = F(b) - F(a)$$

(II) Suppose that $f$ is Continuous on $[a,b]$ and $F(x) = \int_a^x f(t) \, dt$, then $F$ is Continuous on $[a,b]$ and differentiable on $(a,b)$ and $F'(x) = f(x)$.

$$F(x) = \int_a^x f(t) \, dt \Rightarrow F'(x) = f(x)$$

Example: Q4) (b) $\displaystyle\int_0^{\frac{\pi}{6}} (\sec x + \tan x)^2 \, dx$

$$= \int_0^{\frac{\pi}{2}} \sec^2 x + 2\tan x \sec x + \underbrace{\tan^2 x}_{\sec^2 x - 1}) \, dx$$

$$= \int_0^{\frac{\pi}{6}} (2\sec^2 x + 2\tan x \sec x - 1) \, dx$$

$$= (\underbrace{2\tan x + 2\sec x - x}_{F(x)}) \Big|_0^{\frac{\pi}{6}}$$

$$= F\left(\tfrac{\pi}{6}\right) - F(0) = \left(2\tan\tfrac{\pi}{6} + 2\sec\tfrac{\pi}{6} - \tfrac{\pi}{6}\right) - \left(2\tan 0 + 2\sec 0 - 0\right)$$

$$= 2 \cdot \frac{1}{\sqrt{3}} + 2 \cdot \frac{2}{\sqrt{3}} - \frac{\pi}{6} - 2 = \cdots$$

---

Example: Find the derivative of the following

(a) $\displaystyle\frac{d}{dx} \int_0^{x} \boxed{\sin t} \, dt = \sin x$

---

(You can first find the Integration:

$$\int_0^x \sin t \, dt = -\cos t \Big|_0^x = -\cos x + 1$$

Now differentiate : $\left(\int_0^x \sin t \, dt\right)' = (-\cos x + 1)' = \sin x$.

(b) $$\frac{d}{dx}\int_1^{x^2}\frac{dt}{1+t^2} = \frac{1}{1+(x^2)^2}\cdot 2x = \frac{2x}{1+x^4}.$$

In General: $$\left(\int_a^{g(x)} f(t)\,dt\right)' = f(g(x))\cdot g'(x) \quad -\ ①$$

$$\left(\int_{f(x)}^{g(x)} h(t)\,dt\right)' = h(g(x))\cdot g'(x) - h(f(x))\cdot f'(x)$$

Example: $$\frac{d}{dx}\int_{\sin x}^{x^3}\frac{1}{1+t^2}\,dt$$

$$= \frac{1}{1+(x^3)^2}\cdot(3x^2) - \frac{1}{1+(\sin x)^2}\cdot(\cos x).$$

$$= \frac{3x^2}{1+x^6} - \frac{\cos x}{1+\sin^2 x}$$

Example: $$\frac{d}{dx}\left(\int_a^b f(t)\,dt\right) = 0.$$

**Note:** If $f(x) \geq 0$ is an Integrable function on $[a, b]$, then $\int_a^b f(x)\,dx$ is the area enclosed between the Curve $f(x)$ and the $x$-axis.

**Example:** Find the area enclosed between $f(x)$ and the $x$-axis, where $f(x) = 2x\sqrt{x^2+1}$, $x \in [0, 1]$

$$\text{Area} = \int_0^1 \underline{2x}\,\sqrt{x^2+1}\;\underline{dx}$$

Let $u = x^2 + 1$, then $du = 2x\,dx$.

when $x = 0$, then $u = 1$ & when $x = 1$, then $u = 2$.

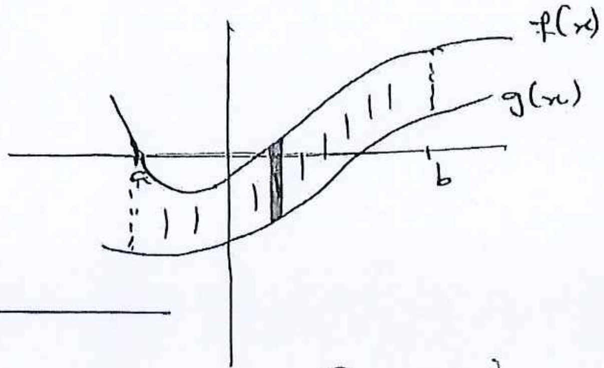
$$\Rightarrow \text{Area} = \int_1^2 \sqrt{u}\,du = \frac{2}{3} u^{3/2} \Big]_1^2$$

$$= \frac{2}{3}\left((2)^{3/2} - (1)^{3/2}\right) = \frac{2}{3}\left(2\sqrt{2} - 1\right) = \frac{4\sqrt{2} - 2}{3}.$$

## Area Between Curves:

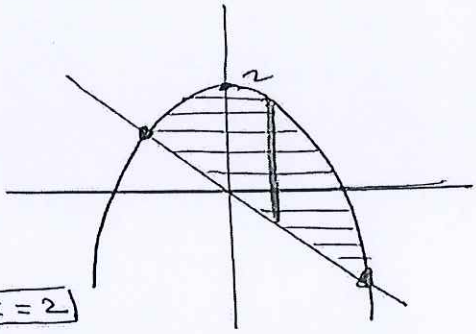
We can find the area enclosed between two functions $f(x)$ and $g(x)$ in some Interval $[a, b]$, where $f(x) \geq g(x)$ using the formula:

$$A = \int_a^b \left( f(x) - g(x) \right) dx$$




**Example**: Find the area enclosed between the Curves:

$f(x) = 2 - x^2$ & $y = -x$.




We find the Intersection points:

$$2 - x^2 = -x \iff x^2 - x - 2 = 0$$

$$\iff (x+1)(x-2) = 0 \iff \boxed{x = -1} \; \& \; \boxed{x = 2}$$

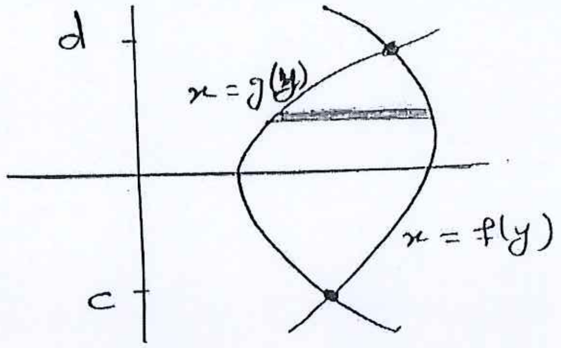
$$\Rightarrow \text{Area} = A = \int_{-1}^{2} (2 - x^2) - (-x)\, dx = \int_{-1}^{2} (2 - x^2 + x)\, dx.$$

$$= 2x - \frac{x^3}{3} + \frac{x^2}{2} \Big]_{-1}^{2} = \left(4 - \frac{8}{3} + \frac{4}{2}\right) - \left(-2 + \frac{1}{3} + \frac{1}{2}\right)$$

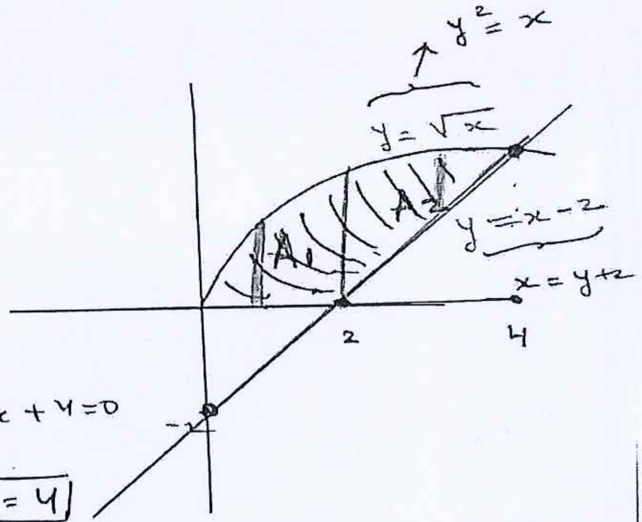
$$= \boxed{4.5}.$$

**Note:** Sometimes, the functions are expressed in terms of $y$ in some Interval $[c,d]$, so the area in this case is

$$A = \int_c^d \big( \underbrace{f(y)}_{\text{right}} - \underbrace{g(y)}_{\text{left}} \big)\, dy.$$




**Example:** Find the area enclosed between $y = \sqrt{x}$, the $x$-axis and the Line $y = x - 2$.




For $y = x - 2$

when $x = 0$, then $y = -2$

$y = 0$, then $x = 2$

Intersection points: $\sqrt{x} = x - 2$

$\Leftrightarrow \quad x = x^2 - 4x + 4 \Leftrightarrow x^2 - 5x + 4 = 0$

$\Leftrightarrow (x-1)(x-4) = 0 \Leftrightarrow \cancel{\boxed{x = 1}}$ & $\boxed{x = 4}$

مرفوضة

$$\Rightarrow \text{Area} = A_1 + A_2 = \int_0^2 \sqrt{x}\, dx + \int_2^4 \left(\sqrt{x} - (x-2)\right) dx.$$

$$= \frac{2}{3}x^{3/2}\Big]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

$$= \cancel{\frac{2}{3}\left(\sqrt{2^3}\right)} + \left(\frac{2}{3}\left(\sqrt{4^3}\right) - \frac{16}{2} + 8\right) - \left(\cancel{\frac{2}{3}\sqrt{2^3}} - \frac{4}{2} + 4\right)$$

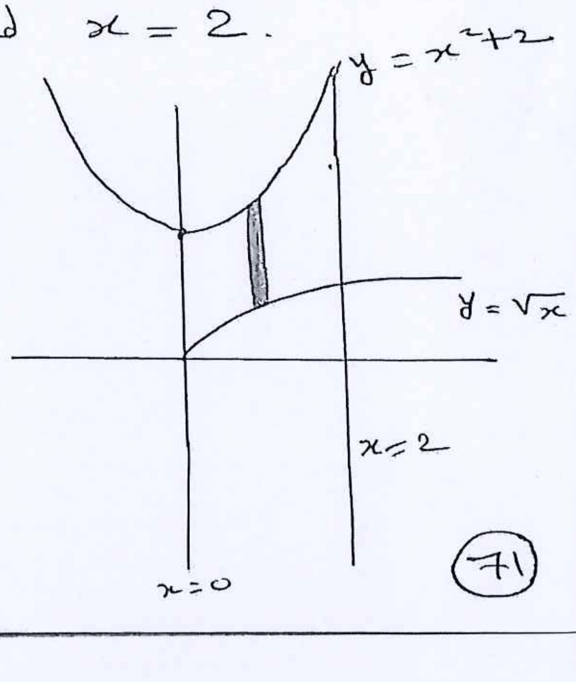
$$= \frac{2}{3}(8) - 8 + 8 + 2 - 4 = \frac{16}{3} - 2 = \boxed{\frac{10}{3}}$$

OR: Area $= \int_0^2 (y+2) - y^2 \, dy = \frac{y^2}{2} + 2y - \frac{y^3}{3}\Big]_0^2 = \boxed{\frac{10}{3}}$

OR: Area $= \int_0^4 (\sqrt{x} - 0)\, dx - \int_2^4 (x-2) - (0)\, dx. = \boxed{\frac{10}{3}}$

Example: Find the area enclosed between the curves

$y = x^2 + 2$, $y = \sqrt{x}$, $x = 0$ and $x = 2$.

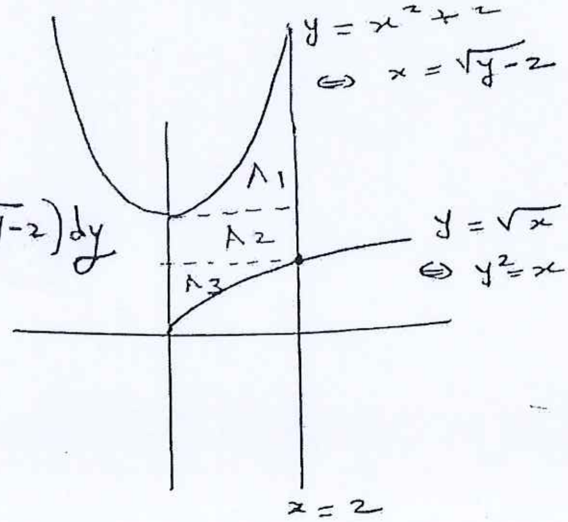



Area $= \int_0^2 (x^2+2) - (\sqrt{x})\, dx$

$$= \frac{x^3}{3} + 2x - \frac{2}{3}x^{3/2}\Big]_0^2$$

$= \ldots$

OR: Area = $A_3 + A_2 + A_1$




$$= \int_0^{\sqrt{2}} (y^2 - 0)\,dy + \int_{\sqrt{2}}^{2} 2\,dy + \int_2^6 (2 - \sqrt{y-2})\,dy$$

$= - - - .$

**Theorem: The Substitution Rule:** If $u = g(x)$ is a differentiable function whose range is $I$, and $f$ is Cont. on $I$, then:

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du$$

**Example:** $\int x^2 \sin(x^3)\,dx$ , Let $u = x^3$

$du = 3x^2\,dx$

$\frac{du}{3} = x^2\,dx$

$$= \int \frac{\sin u}{3}\,du$$

$$= \frac{-1}{3}\cos u + C = -\frac{1}{3}\cos(x^3) + C.$$

Example: $\int \frac{1}{\sqrt{x}} \left( \frac{1}{(1+\sqrt{x})^2} \right) dx$.

Let $u = 1+\sqrt{x}$

$$du = \frac{1}{2\sqrt{x}}\, dx \iff 2\,du = \frac{dx}{\sqrt{x}}$$

$$\Rightarrow \int \frac{2}{u^2}\, du = 2\,\frac{u^{-1}}{-1} + C = \frac{-2}{1+\sqrt{x}} + C.$$