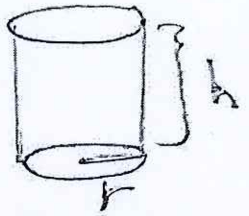
Ch 6 Applications of Definite Integrals:

6.1 Volume Using Cross-Sections:

The volume of the cylindrical solid is always defined to be its base area times its height.

Volume $= \pi r^2 . h$



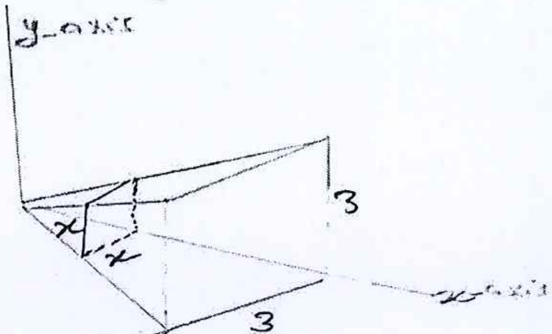
Volume by Slicing:

Def: The volume of a solid of integrable cross-sectional area $A(x)$ from $x=a$ to $x=b$ is $V = \int_a^b A(x)\,dx$.

Calculating the Volume of a Solid:

1. Sketch the solid and a typical cross section.
2. Find a formula of $A(x)$, (Area of Cross-section).
3. Find the limits of Integration.
4. Integrating $A(x)$ Using FTC.

Example: A pyramid 3 m high has a square base that is 3 m on a side. The Cross-section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the Volume of the pyramid.



Cross-sections are squares:

$\Rightarrow \quad A(x) = x^2.$

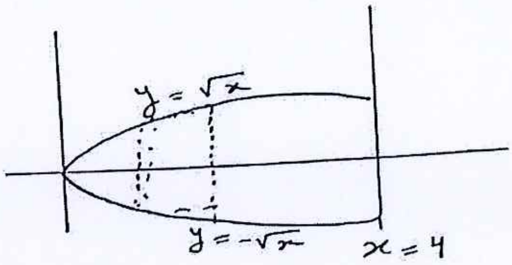
$$\text{Volume} = \int_0^3 A(x)\,dx = \int_0^3 x^2\,dx = \frac{x^3}{3}\Big|_0^3 = \boxed{9} \ m^3.$$

Note: If the Cross-section $\perp$ x-axis, then we Integrate w.r.t x.

If the Cross-section $\perp$ y-axis, then we Integrate w.r.t y.

Example: Find the Volume of a Solid whose cross-sections are squars perpendicular to x-axis between $x = 0$ & $x = 4$. whose side of each square runs from $y = \sqrt{x}$ & $y = -\sqrt{x}$

$$V = \int_a^b A(x)\, dx$$



$$= \int_0^4 (2\sqrt{x})^2\, dx = 32$$

Example: Find the Volume of a solid whose cross-sections are
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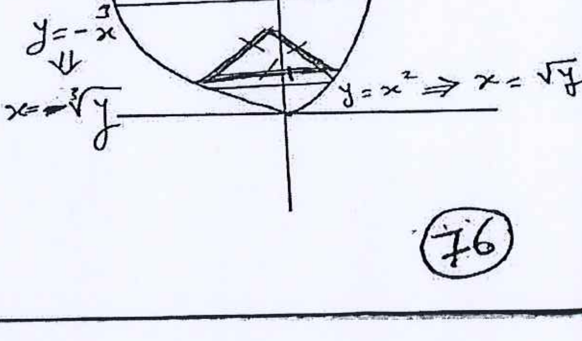
equilateral triangles perpendicular to y-axis between $y = 1$ and $y = 4$ whose side of each triangle runs from $y = -x^3$ & $y = x^2$.

$$V = \int_c^d A(y)\, dy \; , \text{where } A = \frac{1}{2} a.a \sin\frac{\pi}{3}$$

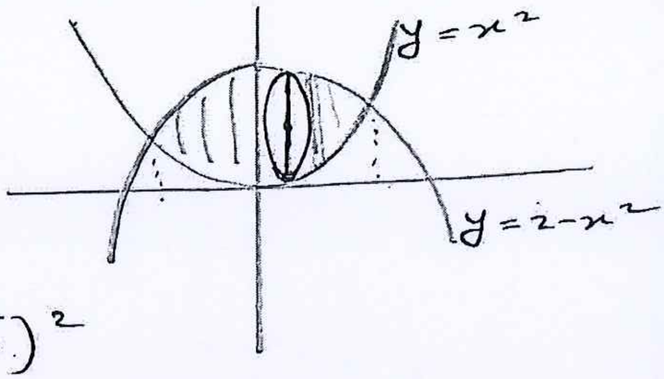


$$= \int_1^4 \frac{\sqrt{3}}{4} a^2\, dy$$

$$= \int_1^4 \frac{\sqrt{3}}{4} \left(\sqrt{y} + \sqrt[3]{y}\right)^2 dy =$$

(12) The Solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$.

The cross-sections perpendicular to the x-axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$.



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Diameter $= (2 - x^2) - x^2$

$= 2 - 2x^2.$

$A(x) = \pi r^2 = \pi \left(\frac{2 - 2x^2}{2} \right)^2$

$= \pi (1 - x^2)^2$

To find the Limits of Integration:

$2 - x^2 = x^2 \iff 2x^2 = 2 \iff x = \pm 1.$

$$\text{Volume} = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} \pi (1 - x^2)^2\,dx$$

$$= \int_{-1}^{1} \pi (1 - 2x^2 + x^4)\,dx = \frac{16}{15}.$$

Solids of Revolution ; The Disk Method

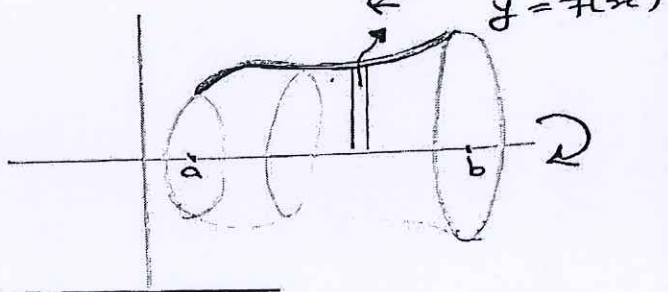
The area of Cross-section $A(x)$ is the area of a disk of radius $R(x)$.

$$A(x) = \pi \left(R(x)\right)^2.$$

Def: Volume by Disks for Rotation about x-axis:

$$V = \int_a^b A(x)\,dx = \int_a^b \pi \left(R(x)\right)^2 dx.$$

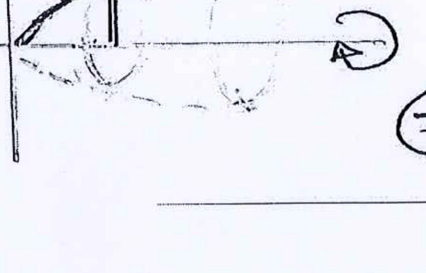
Cross-section $\perp$ axis of Revolution



Example: The region between the Curve $y = \sqrt{x}$ $0 \le x \le 4$ and the x-axis is revolved about the x-axis to generate a solid. Find its Volume.

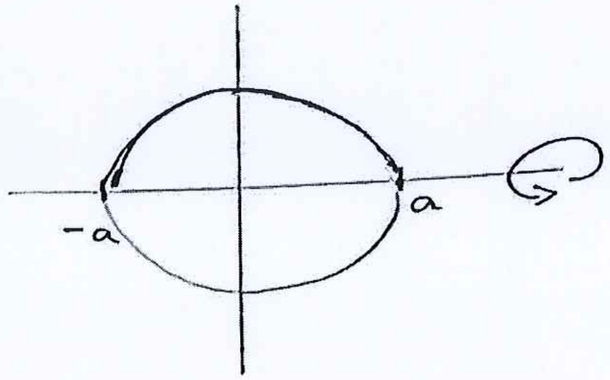
$$V = \int_0^4 \pi \left(R(x)\right)^2 dx = \int_0^4 \pi \left(\sqrt{x}\right)^2 dx$$

$$= \int_0^4 \pi x\,dx = \frac{\pi x^2}{2}\Big|_0^4 = \boxed{8\pi}$$



Example: The circle $x^2+y^2=a^2$ is rotated about the x-axis to generate a sphere.

Find its volume.



$x^2+y^2=a^2 \Leftrightarrow y^2=a^2-x^2$

$\Rightarrow \quad y=+\sqrt{a^2-x^2}.$

$\Rightarrow \quad R(x)=\sqrt{a^2-x^2}.$

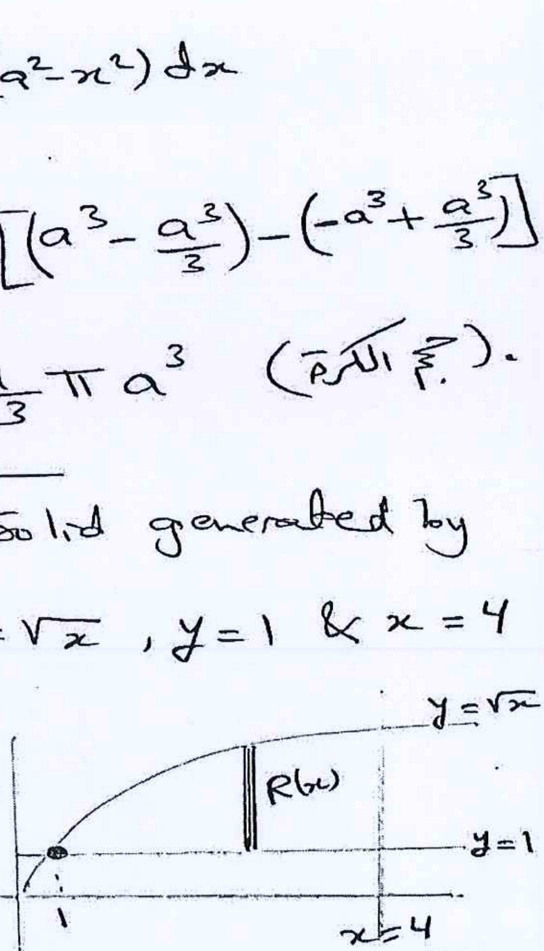
$$\text{Volume} = \int_{-a}^{a} \pi\left(\sqrt{a^2-x^2}\right)^2 dx = \int_{-a}^{a} \pi(a^2-x^2)\,dx$$

$$= \pi\left[a^2x-\frac{x^3}{3}\right]_{-a}^{a} = \pi\left[\left(a^3-\frac{a^3}{3}\right)-\left(-a^3+\frac{a^3}{3}\right)\right]$$

$$= \pi\left(2a^3-\frac{2}{3}a^3\right) = \frac{4}{3}\pi a^3$$ (حجم الكرة).

Example: Find the volume of the solid generated by revolving the region bounded by $y=\sqrt{x}$, $y=1$ & $x=4$ about $y=1$?

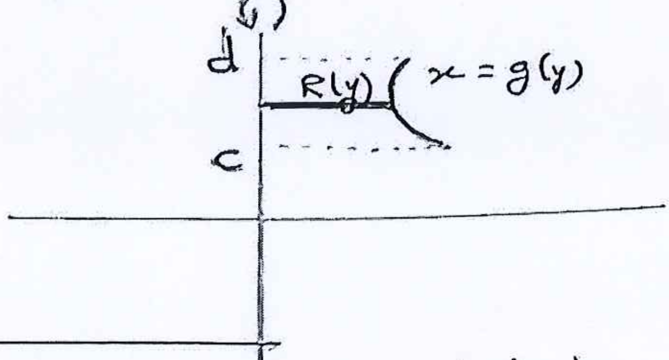
$1=\sqrt{x} \Leftrightarrow x=1$ (point of Intersection)



$$V=\int_{1}^{4}\pi\left(\sqrt{x}-1\right)^2 dx = \frac{7\pi}{6}$$

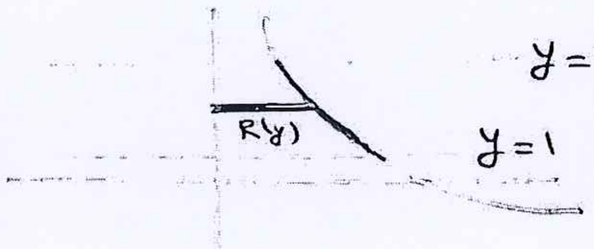
Def: Volume by Disks for Rotation about y-axis

$$V = \int_c^d A(y)\,dy = \int_c^d \pi\,(R(y))^2\,dy.$$



Example: Find the Volume of the Solid generated by revolving the region between the y-axis and the curve $x = \frac{2}{y}$, $1 \le y \le 4$ about y-axis.

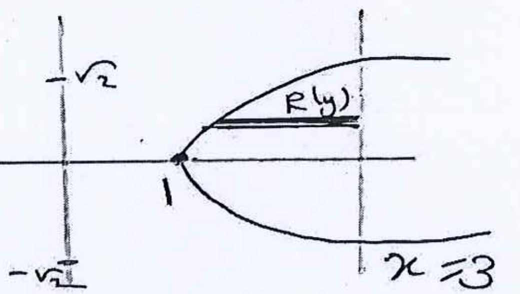
$$V = \int_1^4 \pi \left(\frac{2}{y}\right)^2 dy = 3\pi.$$



Example: Find the Volume of the Solid generated by revolving the region between the parabola $x = y^2+1$ and the Line $x = 3$ about $x = 3$.

$y^2+1 = 3 \Leftrightarrow y = \pm\sqrt{2}$.

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi \left(3 - (y^2+1)\right)^2 dy = \cdots$$

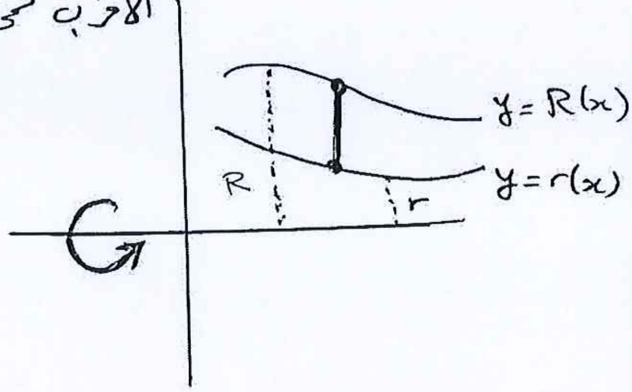


Solids of Revolution : The Washer Method :

Notice the following graph.

$R(x)$: Outer radius - الأبعد عن محور الدوران

$r(x)$: Inner radius - الأقرب لمحور الدوران



In Disk Method : $r(x) = 0$

Def: Volume by Washers for Rotation about x-axis:

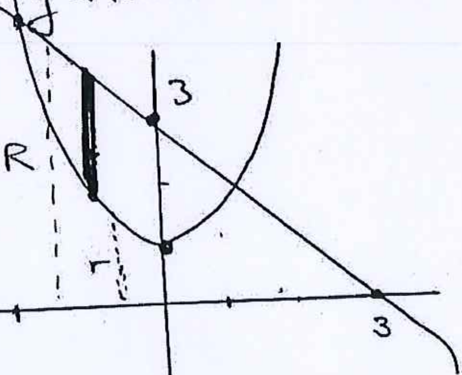
$$V = \int_a^b A(x)\,dx = \int_a^b \pi\left((R(x))^2 - (r(x))^2\right)dx.$$

Example: The region bounded by the Curve

$$y = x^2+1 \quad \text{and the Line} \quad y = -x+3$$

is revolved about the x-axis to generate a Solid.

Find its Volume.



$R(x) = -x+3$, $r(x) = x^2+1$

$$V = \int_{-2}^{1} \pi\left((-x+3)^2 - (x^2+1)^2\right)dx = \boxed{\frac{117\pi}{5}}$$

points of Intersection : $x^2+1 = -x+3 \iff \boxed{x=1}$

Example: The region bounded by the parabola $y = x^2$ and the Line $y = 2x$ in the first (أول) quadrant (الربع) is revolved about (y-axis) to generate a solid. Find the Volume of the Solid.

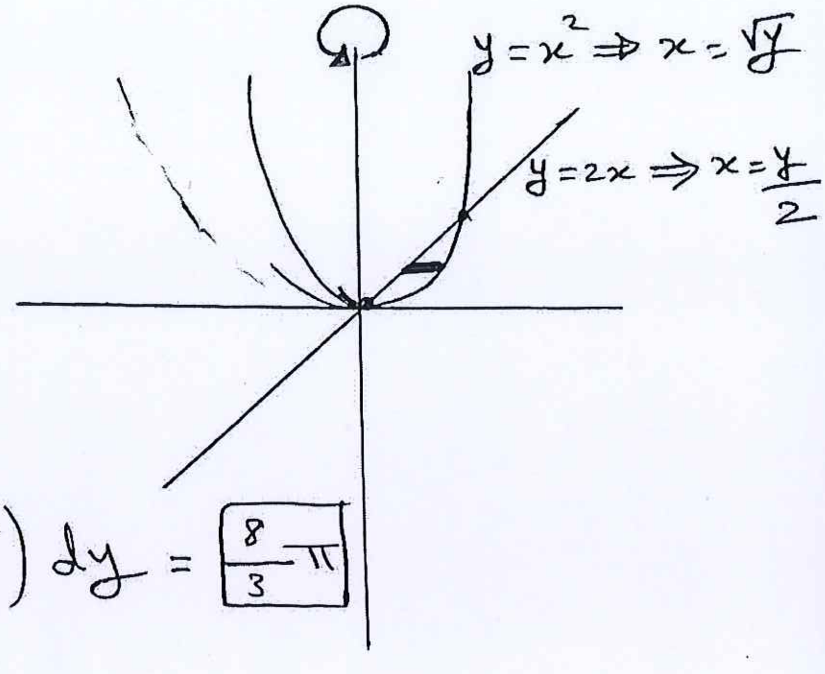
points of Intersection:

$$x^2 = 2x$$

$$\Leftrightarrow \quad x^2 - 2x = 0$$

$$x(x-2) = 0$$

$$\Rightarrow \quad x = 0 \quad \& \quad x = 2$$



$$V = \int_0^4 \pi \left((\sqrt{y})^2 - \left(\frac{y}{2}\right)^2 \right) dy = \boxed{\frac{8}{3}\pi}$$

Lecture Problems:

Q10) Find the Volume of the Solid whose base is the disk $x^2+y^2 \le 1$. The Cross-sections by planes perpendicular to the y-axis between

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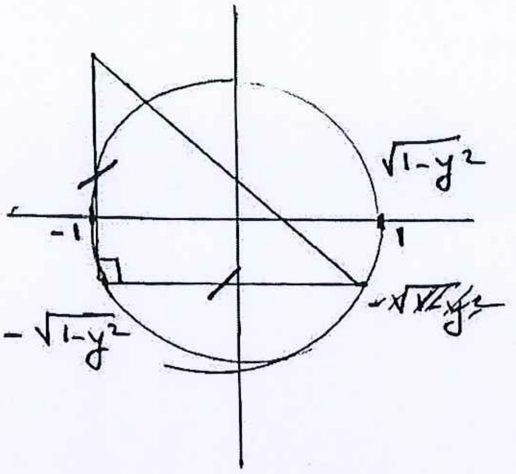
$y=-1$ and $y=1$ are isosceles right triangles with one Leg in the disk

$x^2+y^2=1 \iff x = \pm\sqrt{1-y^2}$.



$$A(y) = \tfrac{1}{2}(\text{base})(\text{height})$$

$$= \tfrac{1}{2}\left[\sqrt{1-y^2}+\sqrt{1-y^2}\right]^2$$

$$= \tfrac{1}{2}\left(2\sqrt{1-y^2}\right)^2$$

$$= \tfrac{4}{2}(1-y^2) = 2(1-y^2)$$

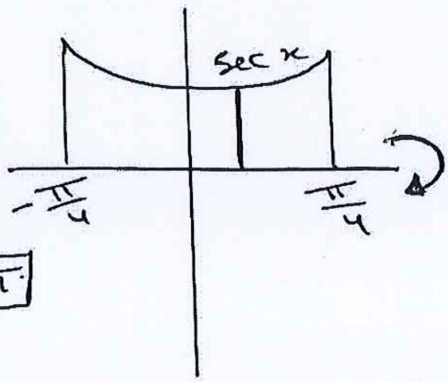
$$V = \int_{-1}^{1} 2(1-y^2)\,dy = \int_{-1}^{1} 2-2y^2\,dy$$

$$= \cancel{\int}\left(2y - \frac{2y^3}{3}\right)\Big]_{-1}^{1} = \boxed{\frac{8}{3}}$$

Q24) Find the volume of the solid for:

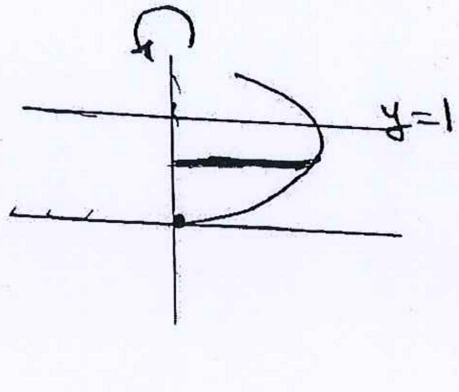
$y = \sec x$, $y = 0$, $x = \frac{-\pi}{4}$, $x = \frac{\pi}{2}$ when revolved about x-axis.

$$V = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \pi \left[\sec^2 x\right] dx$$



$$= \pi . \tan x \Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \pi\left[1 - {}^{-}1\right] = \boxed{2\pi}$$

32) $x = \sqrt{\dfrac{2y}{y^2+1}}$, $x = 0$, $y = 1$ about y-axis.



$$V = \int_0^1 \pi \left(\frac{\sqrt{2y}}{y^2+1}\right)^2 dy$$

$$= \int_0^1 \pi \frac{(2y)}{(y^2+1)^2} dy$$

Using substitution: Let $u = y^2+1 \Rightarrow du = 2y\,dy$

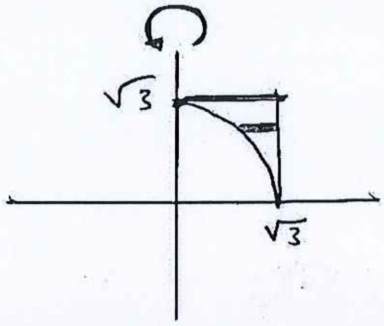
when $y = 0$, then $u = 1$

when $y = 1$, then $u = 2$

$$= \int_1^2 \pi \frac{du}{u^2} = \frac{-1}{u}\Big]_1^2 = \boxed{\frac{\pi}{2}}$$

Q44) The region in the first quadrant bounded on the Left by $x^2+y^2=3$, on the right by $x=\sqrt{3}$ and above by $y=\sqrt{3}$ is revolved about y-axis. Find the Volume.

Using Washer Method, since there is a gap between the region and the axis of Revolution.

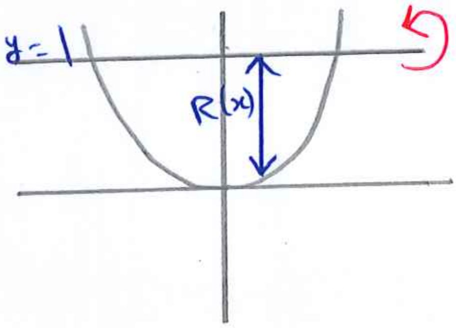


$$\Rightarrow \quad V = \int_0^{\sqrt{3}} \pi \left((\sqrt{3})^2 - \left(\sqrt{3-y^2}\right)^2 \right) dy$$

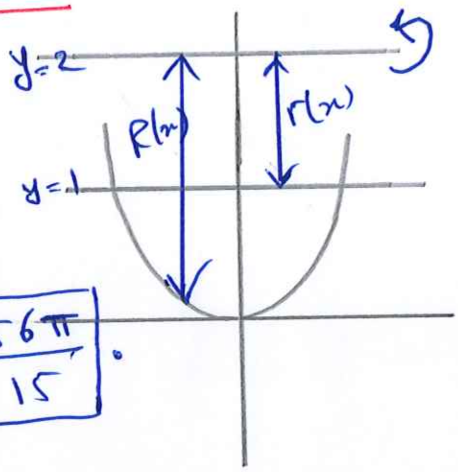
$$= \boxed{\pi(\sqrt{3})}$$

Q(49) Find the Volume of the Solid generated by revolving the region bounded by $y = x^2$, $y = 1$ about :

(a) $y = 1$.

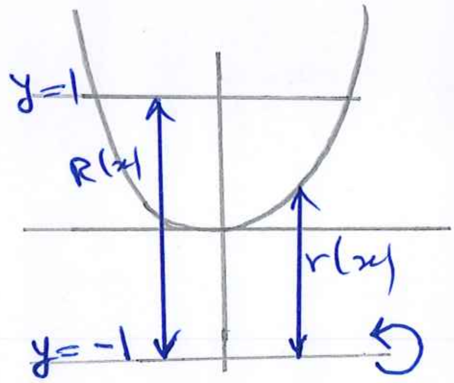


$$v = \int_{-1}^{1} \pi \left(1 - x^2\right)^2 dx = \boxed{\frac{16\pi}{15}}.$$



(b) $y = 2$

$$v = \int_{-1}^{1} \pi \left(\left(2 - x^2\right)^2 - (1)^2\right) dx = \boxed{\frac{56\pi}{15}}.$$



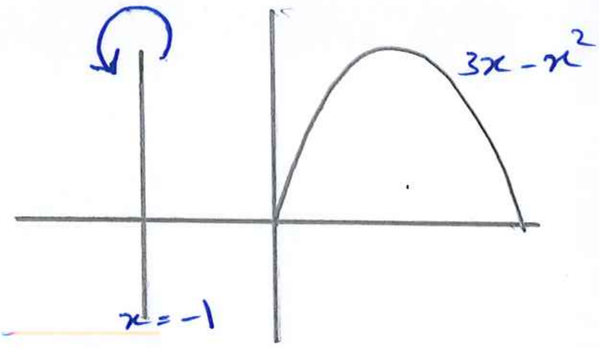
(c) $y = -1$

$$v = \int_{-1}^{1} \pi \left((2)^2 - \left(x^2+1\right)^2\right) dx$$

$$= \boxed{\frac{64\pi}{15}}.$$

6.2 Volumes Using Cylindrical Shells:

Example: The region enclosed by the x-axis and $y = 3x - x^2$ is revolved about $x = -1$ to generate a solid. Find the Volume of the Solid.



Sol: If we want to Find the Volume Using Disk or Washer then we should write $y = f(x)$ as $x = f(y)$ which is not Easy.

Def: vertical line

Shell Formula for Revolution about $x = L$

$$V = \int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dx$$

where: shell radius = Distance between cross-section and axis of Revolution.

shell height = length of the Cross-section.

Note: Cross-section // axis of Revolution
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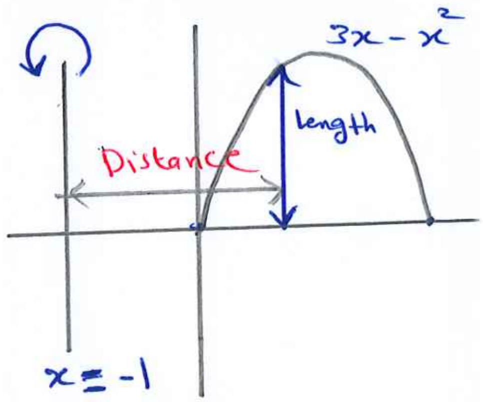
Example: Consider the previous example:

$3x - x^2 = 0 \iff x = 0, 3$

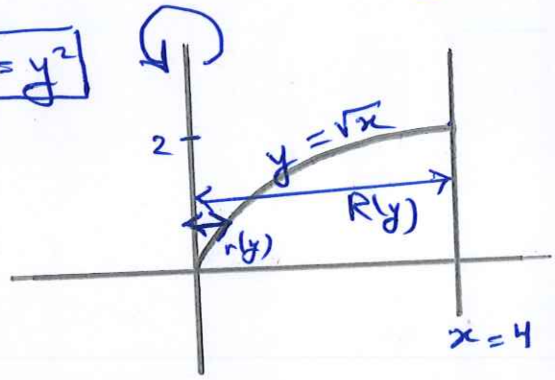


$$V = \int_0^3 2\pi (x+1)(3x - x^2)\, dx$$

$$= \boxed{\frac{45\pi}{2}}.$$

Example: The region bounded by the curve $y = \sqrt{x}$ and the x-axis and the line $x = 4$ is revolved about the y-axis (vertical line) to generate a solid. Find the volume of solid.

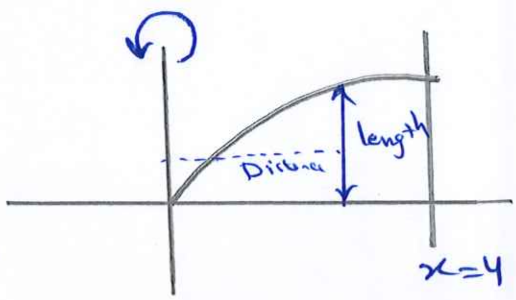
Using Washer Method: $y = \sqrt{x} \iff \boxed{x = y^2}$



$$V = \int_0^2 \pi \left((4)^2 - (y^2)^2 \right) dy$$

$$= \boxed{\frac{128\pi}{5}}.$$

Using Shell Method:



$$V = \int_0^4 2\pi (x)(\sqrt{x})\, dx = \boxed{\frac{128\pi}{5}}.$$

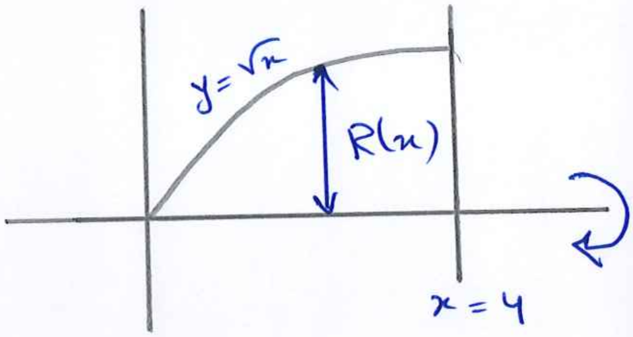
Def: shell formula for Revolution about $y = L$

Horizontal Line

$$V = \int_c^d 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dy$$

Example: The region bounded by the Curve $y = \sqrt{x}$ and x-axis and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the Volume of the Solid.

sol:

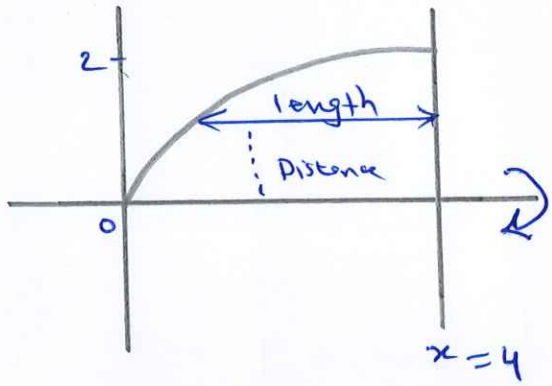


Using Disk Method:

$$V = \int_0^4 \pi (\sqrt{x})^2 dx = \boxed{8\pi}.$$

Using shell Method:

$$y = \sqrt{x} \iff x = y^2$$



$$V = \int_0^2 2\pi (y)(4 - y^2)\, dy$$

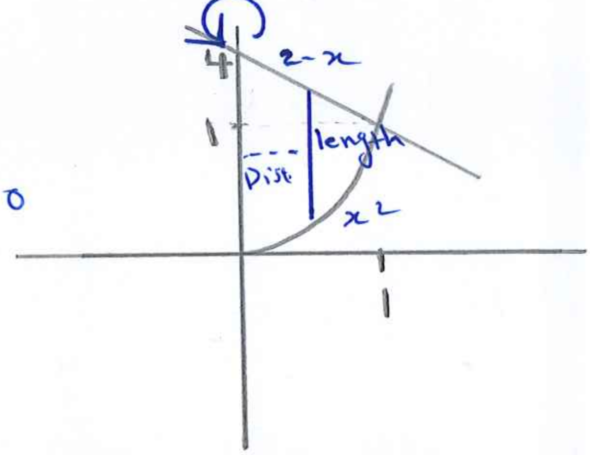
$$= \boxed{8\pi}.$$

Q9 (6.2) Use the shell method to find the Volume of the Solid generated by revolving the region bounded by the Curves $y = x^2$, $y = 2 - x$, $x = 0$, $x \geq 0$ about y-axis

sol: points of Intersection :

$x^2 = 2 - x \iff x^2 + x - 2 = 0$

$\iff \underbrace{x = -2}_{\text{rejected}}$ & $x = 1$



$\Rightarrow \quad V = \int_0^1 2\pi (x)\left((2-x) - x^2\right) dx = \boxed{\frac{5\pi}{6}}.$

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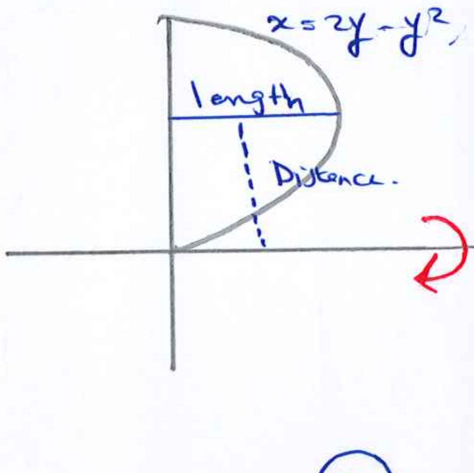
(17) Use shell method to find the Volume of the solid generated by Revolving $x = 2y - y^2$, $x = 0$, about x-axis.

$2y - y^2 = 0 \iff y = 0, 2$

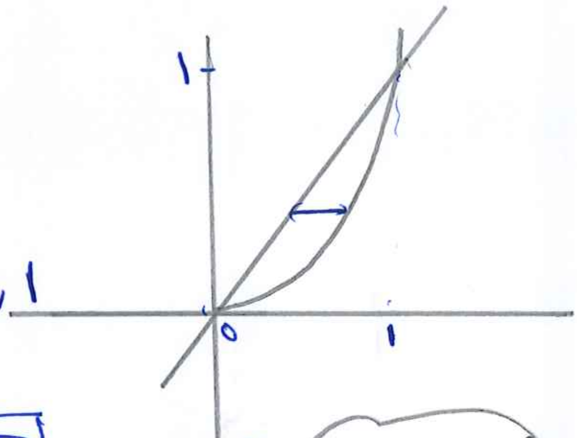
$V = \int_0^2 2\pi (y)(2y - y^2)\, dy$

$= \boxed{\frac{8\pi}{3}}.$



Q29 (6.2) Compute the Volume of the Solid generated by Revolving the region bounded by $y = x$ and $y = x^2$ about each Coordinate axis

(a) Using shell Method.



① $y = x^2 \Rightarrow x = \sqrt{y}$

$y = x \Rightarrow x = y$

$\Rightarrow y = \sqrt{y} \Leftrightarrow y^2 - y = 0 \Leftrightarrow y = 0, 1$

$$V = \int_0^1 2\pi (y)(\sqrt{y} - y)\, dy = \boxed{\frac{2\pi}{15}}$$ about x-axis

② $$V = \int_0^1 2\pi (x)(x - x^2)\, dx = \boxed{\frac{\pi}{6}}$$ about y-axis

(b) Using Washer Method.

① about x-axis:

$$V = \int_0^1 \pi \left((x)^2 - (x^2)^2 \right) dx = \boxed{\frac{2\pi}{15}}.$$

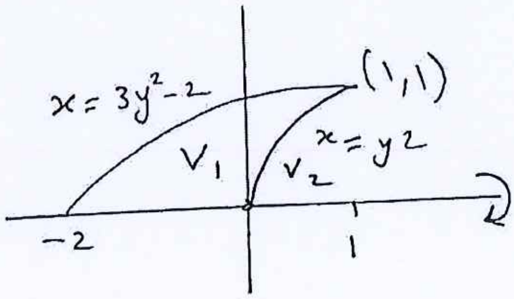
② about y-axis:

$$V = \int_0^1 \pi \left((\sqrt{y})^2 - (y)^2 \right) dy = \boxed{\frac{\pi}{6}}.$$

(39) (6.2) The Region is Revolved about x-axis:

Disk:

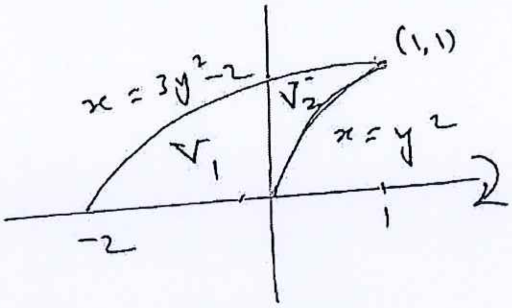
$$V = V_1 - V_2$$



$$V_1 = \int_{-2}^{1} \pi \left[\sqrt{\frac{x+2}{3}}\right]^2 dx = 1.5\pi$$

$$V_2 = \int_{0}^{1} \pi \left[\sqrt{x}\right]^2 dx = 0.5\pi$$

$$V = 1.5\pi - 0.5\pi = \pi$$



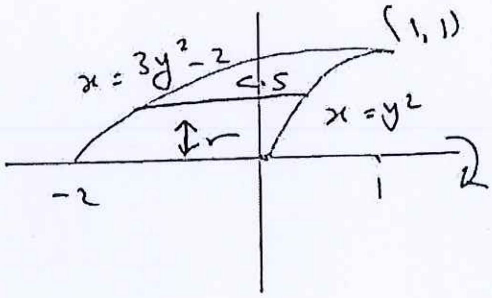
Washer:

$$V = V_1 + V_2$$

$$V_1 = \int_{-2}^{0} \pi\left(\left[\sqrt{\frac{x+2}{3}}\right]^2 - [0]^2\right) dx = \frac{2}{3}\pi$$

$$V_2 = \int_{0}^{1} \pi\left(\left[\sqrt{\frac{x+2}{3}}\right]^2 - \left[\sqrt{x}\right]^2\right) dx = \frac{1}{3}\pi$$

$$V_1 + V_2 = \frac{2}{3}\pi + \frac{1}{3}\pi = \pi$$



Shell

$$V = \int_{0}^{1} 2\pi\,(y)\left[y^2 - (3y^2 - 2)\right] dy = \pi$$

6.3 Arc length.

Length of a Curve $y = f(x)$:

Def: If f' is continuous on $[a, b]$, then the Length (arc length) of the Curve $y = f(x)$ from $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the Integral:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Example: Find the Length of the Curve:

$$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1 \quad , \quad 0 \leq x \leq 1$$

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1 + \left(2\sqrt{2}\, x^{\frac{1}{2}}\right)^2}\, dx$$

$$= \int_0^1 \sqrt{1 + 8x}\, dx = \frac{13}{6}.$$

Dealing with Discontinuities in dy/dx

At some point where dy/dx fails to exist. dx/dy may exist.

Def: If g' is continuous on $[c, d]$, the length of the curve $x = g(y)$ from $A = (g(c), c)$ to $B = (g(b), b)$.

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.$$

Example: Find the length of the curve $y = \left(\frac{x}{2}\right)^{2/3}$ from $x = 0$ to $x = 2$.

$$\frac{dy}{dx} = \frac{1}{3}\left(\frac{2}{x}\right)^{1/3}$$ which is not defined at $x = 0$

Therefore: $x = 2y^{3/2}$ & $\frac{dx}{dy} = 3y^{\frac{1}{2}}$

when $x = 0 \Rightarrow y = 0$, when $x = 2 \Rightarrow y = 1$

$$\Rightarrow L = \int_0^1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{\underbrace{1 + 9y}_{u}}\, dy$$

$du = 9dy$

$$= \int_1^{10} \frac{\sqrt{u}}{9}\, du = \frac{2}{27}\left(10\sqrt{10} - 1\right)$$

The Differential formula for Arc Length.

The arc length function of the curve $y = f(x)$ starting from the point $P_0(a, f(a))$ is defined by:

$$S(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt.$$

Example: Find the arc length function for $f(x) = \frac{x^3}{12} + \frac{1}{x}$ starting from $A = (1, 13/12)$.

$$f'(x) = \frac{x^2}{4} - \frac{1}{x^2} \Rightarrow (f'(x))^2 = \frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}$$

$$1 + (f'(x))^2 = \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2$$

$$\Rightarrow S(x) = \int_1^x \sqrt{\left(\frac{t^2}{4} + \frac{1}{t^2}\right)^2}\, dt = \int_1^x \left(\frac{t^2}{4} + \frac{1}{t^2}\right) dt$$

$$= \left[\frac{t^3}{12} - \frac{1}{t}\right]_1^x = \frac{x^3}{12} - \frac{1}{x} + \frac{11}{12}.$$

If we want to compute the arc length from $A = (1, 13/12)$ to $B = (4, 67/12)$, we simply substitute

$S(4) = 6$

6.3 Q4 Lecture Find the length of the Curve:

$$x = \frac{y^{\frac{3}{2}}}{3} - y^{\frac{1}{2}} \quad \text{from } y = 1 \text{ to } y = 9$$

$$\frac{dx}{dy} = \frac{1}{2} y^{\frac{1}{2}} - \frac{1}{2} y^{-\frac{1}{2}} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{4}\left(y - 2 + \frac{1}{y}\right)$$

$$\Rightarrow L = \int_1^9 \sqrt{1 + \frac{1}{4}\left(y - 2 + \frac{1}{y}\right)}\, dy = \int_1^9 \sqrt{\frac{1}{4}y + \frac{1}{2} + \frac{1}{4y}}\, dy$$

$$= \int_1^9 \sqrt{\left(\frac{1}{2}\sqrt{y} + \frac{1}{2\sqrt{y}}\right)\left(\frac{1}{2}\sqrt{y} + \frac{1}{2\sqrt{y}}\right)}\, dy = \int_1^9 \frac{1}{2}\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right) dy = \boxed{\frac{32}{3}}$$

Lecture

Q19 (a) Find a Curve through (1,1) whose length integral is $L = \int_1^4 \sqrt{1 + \frac{1}{4x}}\, dx$.

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4x} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}, \text{ then}$$

$$y = \sqrt{x} + C.$$

Now $(1,1) \Rightarrow 1 = 1 + C \Rightarrow C = 0$

then $y = \sqrt{x}$ from $(1,1)$ to $(4,2)$

(b) How many such Curves are there?

only one.

6.4 Areas of Surfaces of Revolution:

Def: (Revolution about the x-axis) If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the area of the surface generated by revolving the graph of $y = f(x)$ about the x-axis is

$$S = \int_a^b \underbrace{2\pi y}_{} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

بدونه بكون
arc length

Example: Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$ about x-axis

$$S = \int_1^2 2\pi(2\sqrt{x}) \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}\, dx =$$

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$$= \int_1^2 2\pi(2\sqrt{x}) \cdot \frac{\sqrt{x+1}}{\sqrt{x}}\, dx = \int_1^2 4\pi\sqrt{x+1}\, dx$$

$$= 4\pi \cdot \frac{2}{3}(x+1)^{3/2} \Big]_1^2 = \frac{8\pi}{3}\left(3\sqrt{3} - 2\sqrt{2}\right).$$

Def: Revolution about the y-axis:

If $x = g(y) \geq 0$ is continuously differentiable on $[c,d]$, the area of the surface generated by revolving the graph of $x = g(y)$ about the y-axis is:

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Example: The line segment $x = 1-y$ is revolved about y-axis to generate a cone. Find its

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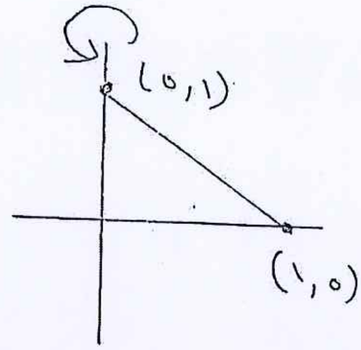
Lateral surface area.



$$S = \int_0^1 2\pi (1-y)\sqrt{1+(-1)^2}\, dy$$

$$= \int_0^1 2\pi (1-y)\sqrt{2}\, dy = \pi\sqrt{2}.$$

(6.4) (15) Find the area of the surface generated by revolving the curve $y = \sqrt{2x - x^2}$ about x-axis, $0.5 \le x \le 1.5$

sol: $\frac{dy}{dx} = \frac{1}{2} \cdot \frac{(2-2x)}{\sqrt{2x-x^2}} = \frac{1-x}{\sqrt{2x-x^2}}$

$$\left(\frac{dy}{dx}\right)^2 = \frac{(1-x)^2}{2x-x^2}$$

$$S = \int_{0.5}^{1.5} 2\pi \left(\sqrt{2x-x^2}\right)\left(\sqrt{1+\frac{(1-x)^2}{2x-x^2}}\right) dx = 2\pi \int_{0.5}^{1.5} dx = \boxed{2\pi}$$

(24) Write an Integral for the area of the surface generated by revolving the curve $y = \cos x$ $-\pi/2 \le x \le \pi/2$ about the x-axis.

$$\frac{dy}{dx} = -\sin x \Rightarrow \left(\frac{dy}{dx}\right)^2 = \sin^2 x$$

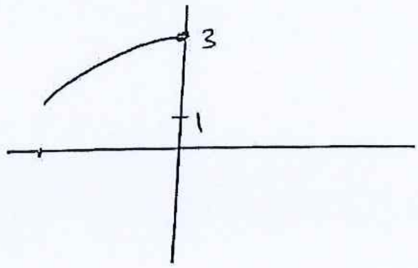
$$S = 2\pi \int_{-\pi/2}^{\pi/2} (\cos x)\sqrt{1+\sin^2 x}\, dx$$

Lecture

6.4 (18) Find the area of the surface generated by revolving the curve

$$x = \frac{1}{3} y^{\frac{3}{2}} - y^{\frac{1}{2}}, \quad 1 \le y \le 3 \quad \text{about } y\text{-axis}$$

Note: to get positive area, we take



$$x = -\left(\frac{1}{3} y^{\frac{3}{2}} - y^{\frac{1}{2}}\right).$$

$$\Rightarrow \frac{dx}{dy} = -\frac{1}{2}\left(y^{\frac{1}{2}} - y^{-\frac{1}{2}}\right) \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{4}\left(y - 2 + y^{-1}\right).$$

$$\Rightarrow S = -\int_1^3 2\pi \left(\frac{1}{3} y^{\frac{3}{2}} - y^{\frac{1}{2}}\right) \sqrt{1 + \frac{1}{4}\left(y - 2 + y^{-1}\right)}\, dy$$

$$= -2\pi \int_1^3 \left(\frac{1}{3} y^{\frac{3}{2}} - y^{\frac{1}{2}}\right) \sqrt{\frac{1}{4}\left(y + 2 + y^{-1}\right)}\, dy$$

$$= -2\pi \int_1^3 \left(\frac{1}{3} y^{\frac{3}{2}} - y^{\frac{1}{2}}\right) \cdot \frac{1}{2} \sqrt{\left(y^{\frac{1}{2}} + y^{-\frac{1}{2}}\right)^2}\, dy$$

$$= \frac{-2\pi}{2} \int_1^3 \left(\frac{1}{3} y^{\frac{3}{2}} - y^{\frac{1}{2}}\right) \cdot \left(y^{\frac{1}{2}} + y^{-\frac{1}{2}}\right) dy.$$

$$= -\pi \int_1^3 y^{\frac{1}{2}} \left[\frac{1}{3} y - 1\right]\left[y^{\frac{1}{2}} + y^{-\frac{1}{2}}\right] dy.$$

$$= -\pi \int_1^3 \left(\frac{1}{3} y - 1\right)\left(y + 1\right) dy = \Rightarrow$$

$$-\pi \int_1^3 \left(\frac{1}{3} y^2 - \frac{2}{3} y - 1\right) dy \qquad \boxed{\frac{16\pi}{3}}$$

(100)

Lecture

6.4 (20) Find the area of the surface generated by revolving the curve $x = \sqrt{2y-1}$, $\frac{5}{8} \le y \le 1$, y-axis

$$\frac{dx}{dy} = \frac{1(2)}{2\sqrt{2y-1}} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{2y-1}.$$

$$S = \int_{5/8}^{1} 2\pi \sqrt{2y-1} \cdot \sqrt{1 + \frac{1}{2y-1}}\, dy$$

$$= \int_{5/8}^{1} 2\pi \sqrt{2y-1} \cdot \frac{\sqrt{2y}}{\sqrt{2y-1}}\, dy.$$

$$= \int_{5/8}^{1} 2\pi \cdot \sqrt{2} \cdot y^{\frac{1}{2}}\, dy = \frac{\pi}{12}\left(16\sqrt{2} - 5\sqrt{2}\right).$$