$\boxed{1.2}$ Combining functions; shifting and scaling Graphs

* If f and g are functions, then for every $x \in D(f) \cap D(g)$, we define:

operation of addition of functions → addition of real numbers

$$(f+g)(x) = f(x) + g(x) \quad \text{"sum"}$$

$$(f-g)(x) = f(x) - g(x) \quad \text{"Difference"}$$

$$(fg)(x) = f(x)\, g(x) \quad \text{"product"}$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \; g(x) \neq 0 \quad \text{"Quotient"}$$

$$(cf)(x) = c f(x), \; c \in \mathbb{R} \quad \text{"Multiply by constant"}$$

Example: If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x}$, then

$(2f)(x) = 2f(x) = 2\sqrt{x}$ with domain $[0,\infty)$

$(f+g)(x) = f(x) + g(x) = \sqrt{x} + \sqrt{1-x}$ with domain $[0,\infty) \cap (-\infty, 1] = [0,1]$

$(f-g)(x) = \sqrt{x} - \sqrt{1-x}$ with domain $[0,1]$

$(f \cdot g)(x) = \sqrt{x}\,\sqrt{1-x} = \sqrt{x(1-x)}$ with domain $[0,1]$

$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x}}{\sqrt{1-x}} = \sqrt{\frac{x}{1-x}}$ with domain $[0,1)$

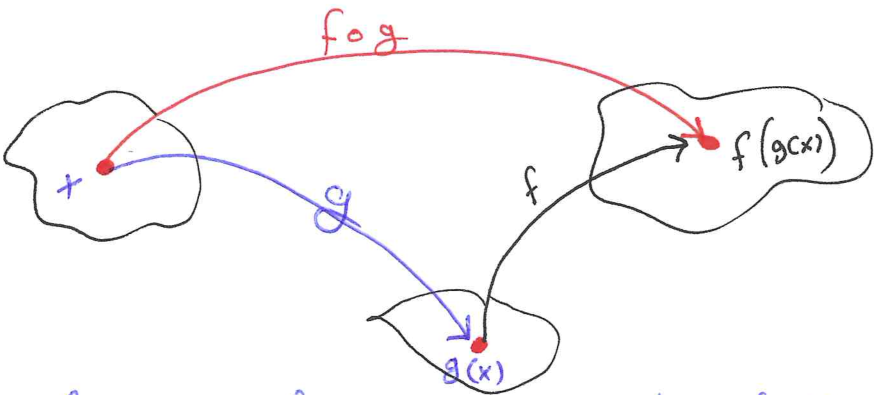
$\left(\frac{g}{f}\right)(x) = \frac{\sqrt{1-x}}{\sqrt{x}} = \sqrt{\frac{1-x}{x}}$ with domain $(0,1]$

Composite functions

Def: Let f and g be two functions. The composite function $f \circ g$ "f composed with g" is defined by

$$(f \circ g)(x) = f\big(g(x)\big)$$

$$R(g) \subset D(f)$$



The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.

Example If $f(x) = \sqrt{x}$ and $g(x) = x + 1$ find

(a) $(f \circ g)(x) = f\big(g(x)\big) = f\big(x+1\big) = \sqrt{x+1}$, $D = [-1, \infty)$

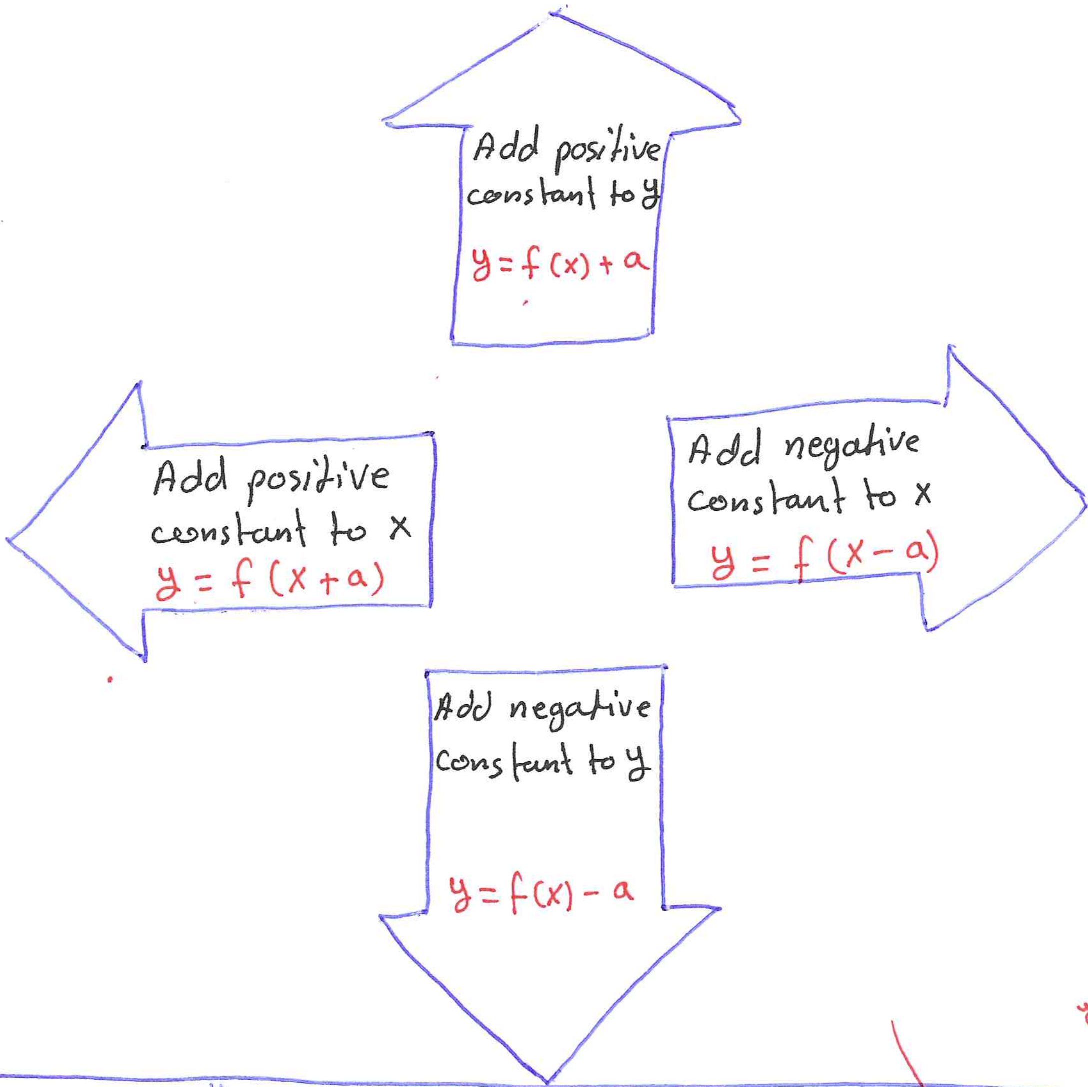
(b) $(g \circ f)(x) = g\big(f(x)\big) = g\big(\sqrt{x}\big) = \sqrt{x} + 1$, $D = [0, \infty)$

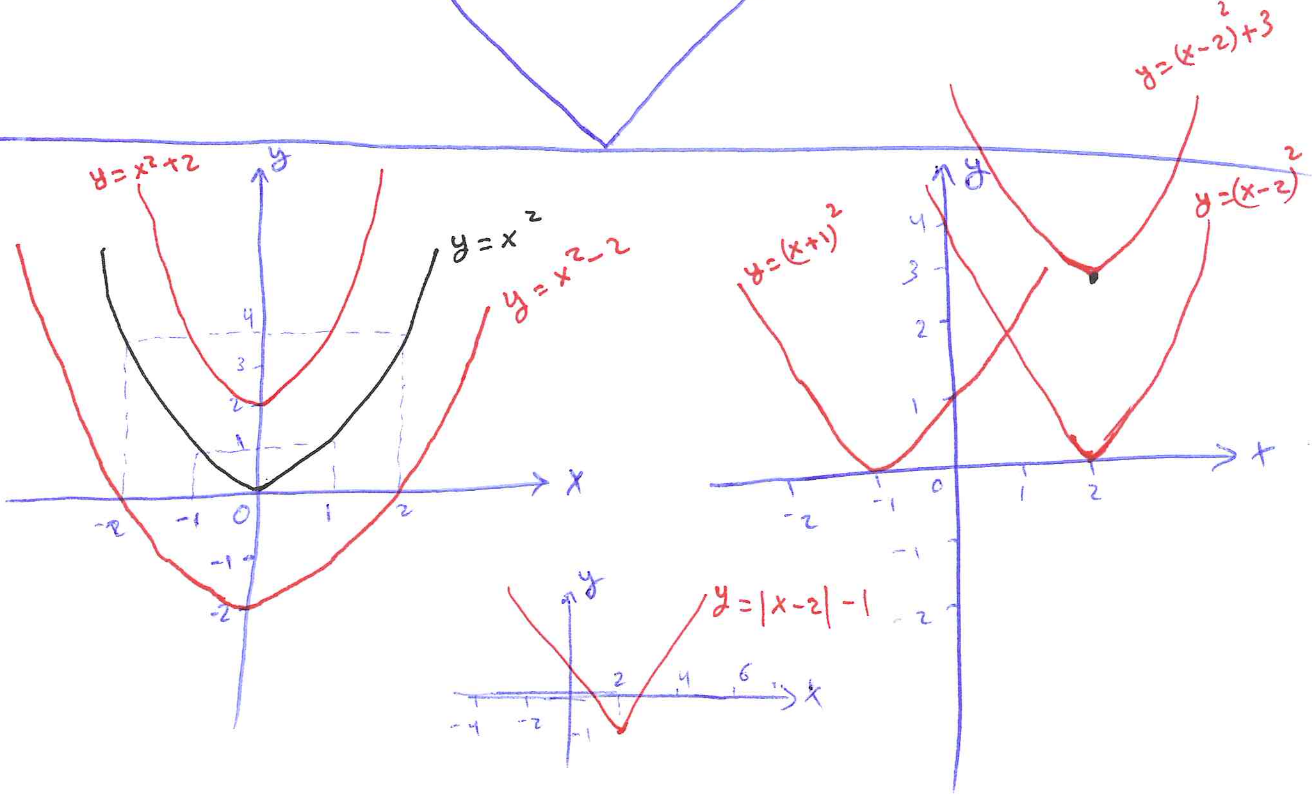
(c) $(f \circ f)(x) = f\big(f(x)\big) = f\big(\sqrt{x}\big) = \sqrt{\sqrt{x}} = x^{\frac{1}{4}}$, $D = [0, \infty)$

(d) $(g \circ g)(x) = g\big(g(x)\big) = g\big(x+1\big) = x + 1 + 1 = x + 2$, $D = \mathbb{R}$

Shifting Graphs

* Horizontal and Vertical shift: Let $a > 0$ and $y = f(x)$

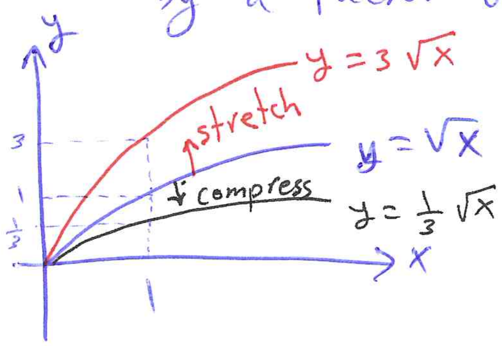




Vertical scaling:

If $c > 1$, then

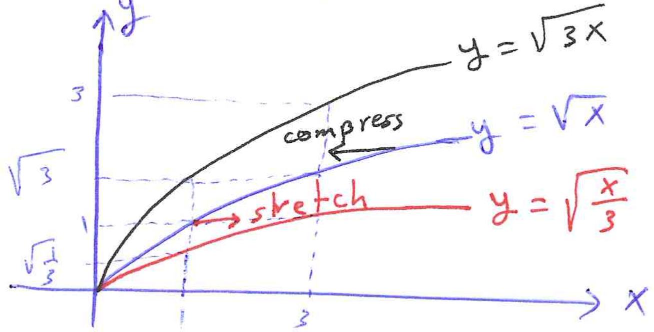
- $y = c\,f(x)$ stretches the graph of f vertically by a factor of c. تمدد
- $y = \frac{1}{c} f(x)$ compresses the graph of f vertically by a factor of c. ضغط



Horizontal scaling

If $c > 1$, then

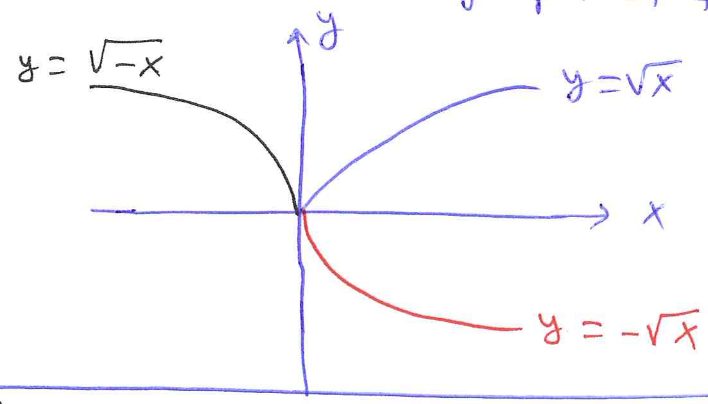
- $y = f(cx)$ compresses the graph of f horizontally by a factor of c
- $y = f\left(\frac{x}{c}\right)$ stretches the graph of f horizontally by a factor of c.



Reflections

* If $c = -1$, then

- $y = -f(x)$ reflects the graph of f across the x-axis
- $y = f(-x)$ reflects the graph of f across the y-axis



Example: Let $f(x) = x^4 - 2x^3 + 1$. Find formulas to

(a) compress the graph horizontally by a factor of 2 followed by a reflection across the y-axis.

compress horizontally by 2 $\Rightarrow f(2x)$

reflection across y-axis $\Rightarrow f(-2x)$

$$f(-2x) = (-2x)^4 - 2(-2x)^3 + 1$$
$$= 16x^4 + 16x^3 + 1$$

(b) compress the graph vertically by a factor 2 followed by a reflection across the x-axis.

compress the graph vertically by 2 $\Rightarrow \frac{1}{2} f(x)$

reflection across x-axis $\Rightarrow -\frac{1}{2} f(x)$

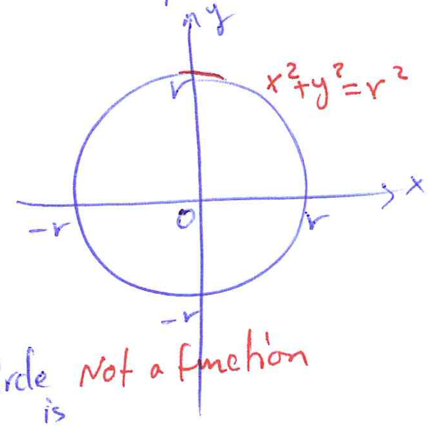
$$-\frac{1}{2} f(x) = -\frac{1}{2}\left[x^4 - 2x^3 + 1\right]$$
$$= -\frac{1}{2}x^4 + x^3 - \frac{1}{2}$$

Ellipses القطع الناقص

* A standard equation for the circle of radius r and centered at point (h, k) is

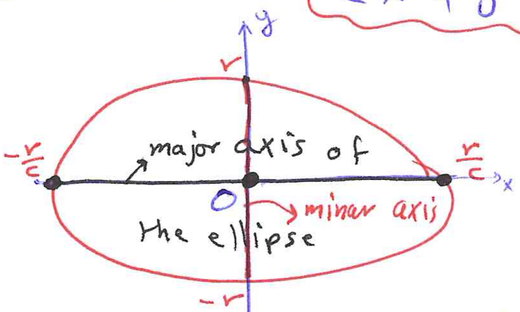
$$(x-h)^2 + (y-k)^2 = r^2$$

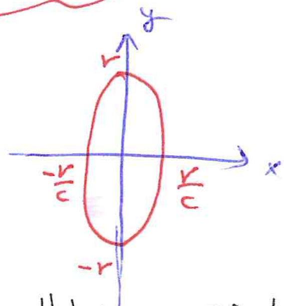
* Circle centered at origin has the following equation $\quad x^2+y^2=r^2 \quad \text{-----} \; ①$



circle is Not a function

Substitute cx for x in equation ① we get $\boxed{c^2x^2+y^2=r^2}$ ✱





y-intercept is the same $r, -r$ in the three figures

• Major axis is the longer line segment.

① • ellipse: $0 < c < 1$
② • not a function
③ • stretches the circle

① • ellipses $c > 1$
② • not a function
③ • compressed horizontally

④ • Major axis is the line segment joining the points $(-\frac{r}{c}, 0)$ and $(\frac{r}{c}, 0)$

⑤ • Minor axis is the line segment joining the points $(0, -r)$ and $(0, r)$

④ • Major axis is the line segment joining the points $(0, -r)$ and $(0, r)$

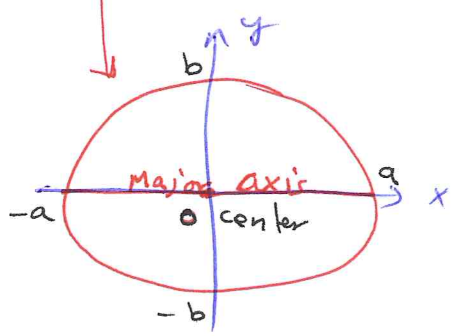
⑤ • Minor axis is the line segment joining the points $(-\frac{r}{c}, 0)$ and $(\frac{r}{c}, 0)$

Divid $\ast$ by r^2 $\Rightarrow$ $\boxed{\frac{c^2x^2}{r^2} + \frac{y^2}{r^2} = 1} \rightarrow (2)$

Take $a = \frac{r}{c}$ and $b = r$

Equation (2) becomes: $\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$ → Ellipse centered at origin

- If $a > b$, then the major axis is horizontal
- If $a < b$, then " " " vertical



$a > b$

The standard equation of an ellipse centered at (h,k) is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$