1.3 Trigonometric Functions

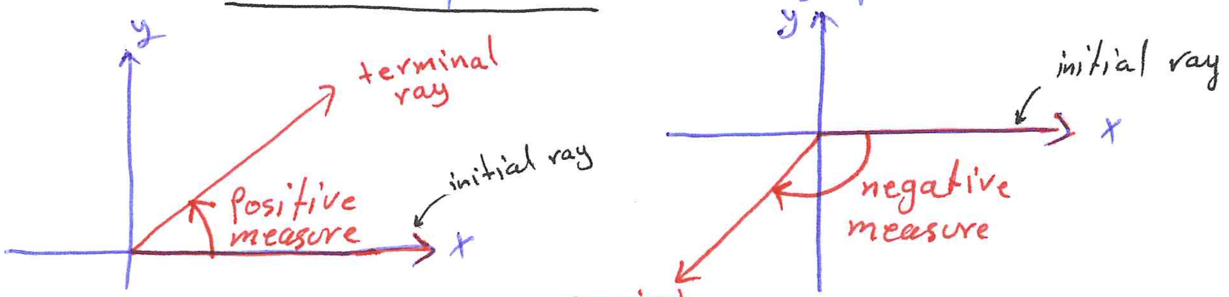
* Angles are measured either by degrees or radians.

$$\boxed{\pi \text{ radians} = 180^\circ}$$

1 radian $= \frac{180}{\pi} \approx 57^\circ$

$1^\circ = \frac{\pi}{180} \approx 0.02$ rad

* Angles in standard position in the xy-plane



counterclockwise　　　　　clockwise
عكس عقارب الساعة　　　　　مع عقارب الساعة

⇒ An angle in the xy-plane is in standard position if its vertex lies at orgin an its initial ray lies along the positive x-axis.

* Conversion formulas:

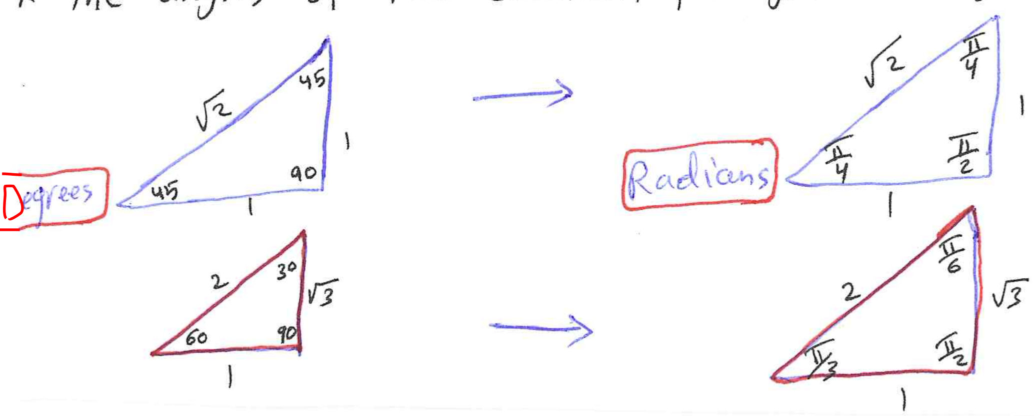
Degrees to radian : multiply by $\frac{\pi}{180}$

Radians to Degrees : multiply by $\frac{180}{\pi}$

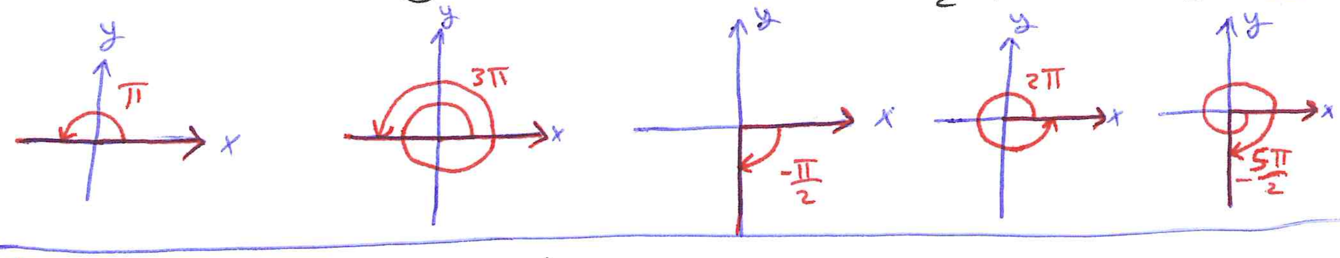
Examples : Convert 45° to radians : $45^\circ \times \frac{\pi}{180} = \frac{\pi}{4}$ rad

Convert $\frac{\pi}{6}$ rad to degrees : $\frac{\pi}{6} \times \frac{180}{\pi} = 30^\circ$

* The angles of two common triangles in degrees and radians :



* Draw the following Angles: π, 3π, $-\frac{\pi}{2}$, 2π, $-\frac{5\pi}{2}$ 18

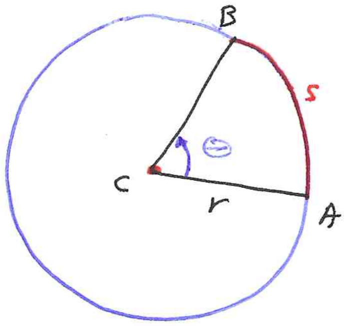


* Radian Measure and Arc Length:

Let s be the arc length AB of a circle of radius r.

The angle ACB is θ measured in radians.

$$\boxed{s = r\theta}$$



* The unit cicle has arc length $s = \theta$

Example: Consider a circle of radius 8

(a) Find the central angle محوطة/ مشدودة subtended by an arc of length 2π

(b) Find the length of an arc subtending a central angle of $\frac{3\pi}{4}$

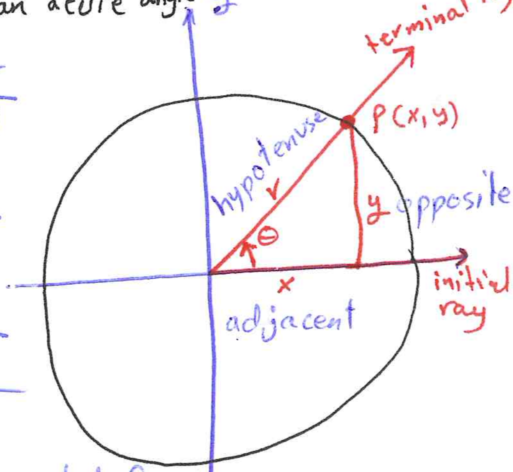
(a) $\theta = \frac{s}{r} = \frac{2\pi}{8} = \frac{\pi}{4}$

(b) $s = r\theta = 8\left(\frac{3\pi}{4}\right) = 6\pi$

obtuse: منفرجة

* The Six Basic Trigonometric Functions of an acute حادة angle

Sine: $\sin\theta = \frac{y}{r}$	Cosecant: $\csc\theta = \frac{r}{y}$
Cosine: $\cos\theta = \frac{x}{r}$	Secant: $\sec\theta = \frac{r}{x}$
Tangent: $\tan\theta = \frac{y}{x}$	Cotangent: $\cot\theta = \frac{x}{y}$



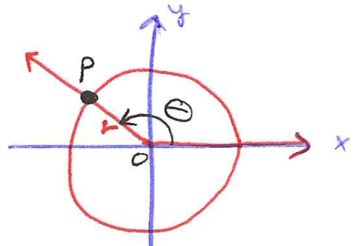
* Note that when $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots$ we have $x = 0$ and so $\tan\theta$ and $\sec\theta$ are not defined.

* Note that when $\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \dots$ we have $y = 0$ and so $\cot\theta$ and $\csc\theta$ are not defined.

⟹ Note also that $x = r\cos\theta$ and $y = r\sin\theta$

$$P(x,y) = (r\cos\theta,\ r\sin\theta)$$

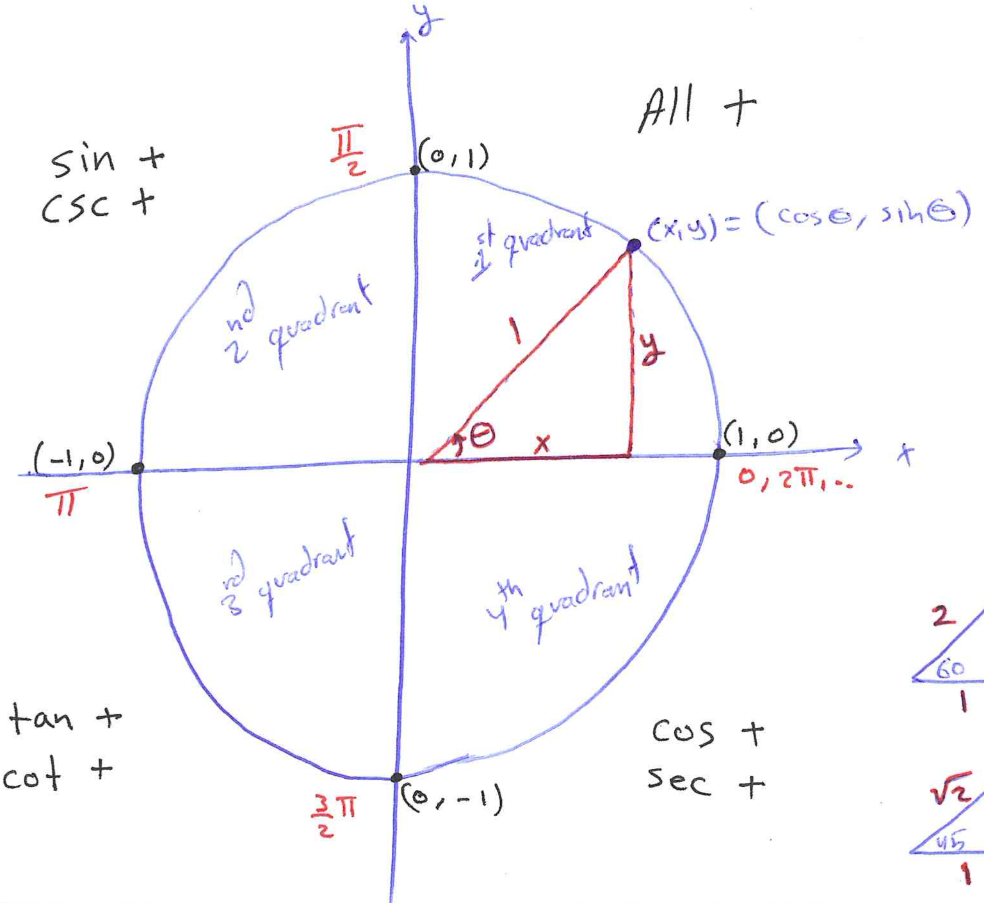
Note that in Unit circle $(x,y) = (\cos\theta, \sin\theta)$

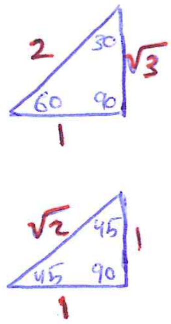


$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}$$

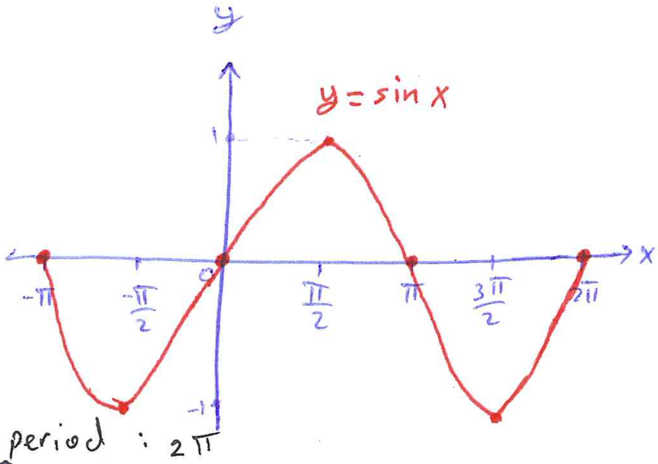
Unit Circle



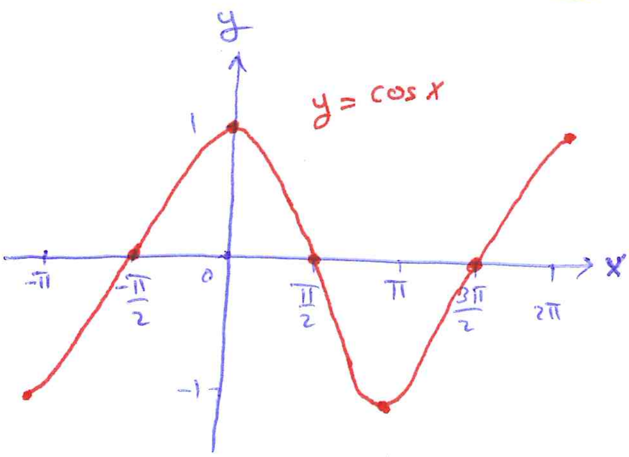


Degrees	30	60	45	0	90	180	270	360	-45	-135	135	-180
θ (radians)	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$-\frac{\pi}{4}$	$-\frac{3\pi}{4}$	$\frac{3\pi}{4}$	$-\pi$
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	0	1	0	-1	0	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	1	0	-1	0	1	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1
$\tan\theta$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	1	0		0		0	-1	1	-1	0

Graphs of Trigonometric Functions



period : 2π
Domain = $(-\infty, \infty)$
Range = $[-1, 1]$



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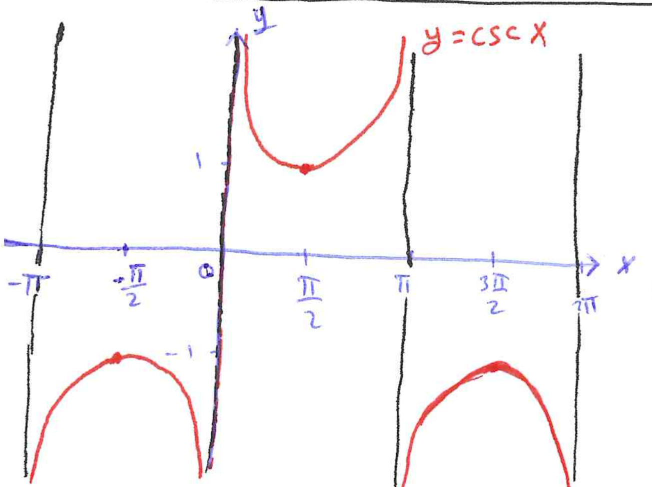
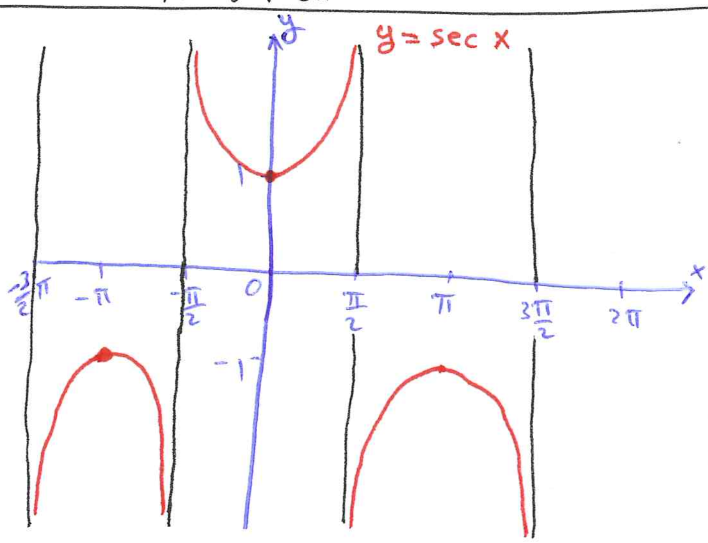
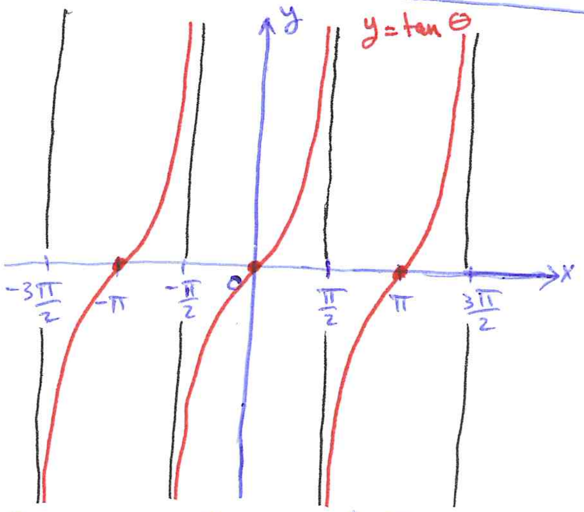


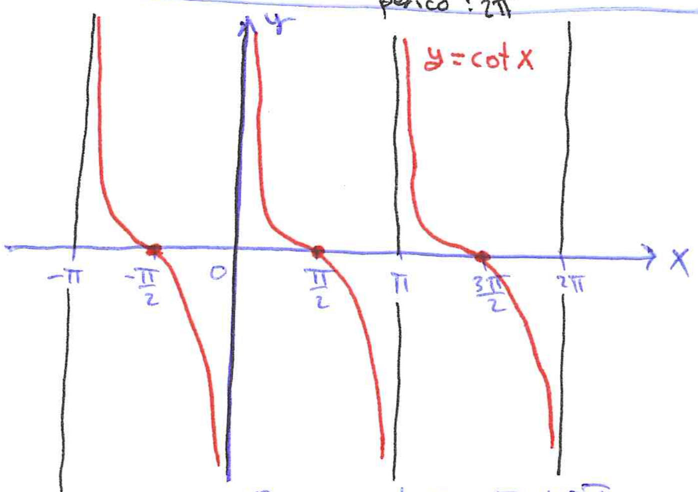
Domain = $\mathbb{R}$ except $0, \pm\pi, \pm 2\pi, \ldots$
Range = $(-\infty, -1] \cup [1, \infty)$ period : 2π



Domain = $\mathbb{R}$ except $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range = $(-\infty, -1] \cup [1, \infty)$
period : 2π



Domain = $\mathbb{R}$ except $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range = $(-\infty, \infty)$ period : π



Domain = $\mathbb{R}$ except $0, \pm\pi, \pm 2\pi, \ldots$
Range = $(-\infty, \infty)$ period : π

Periodicity

* **Definition**: A function $f(x)$ is periodic if there is a positive number p s.t $f(x+p) = f(x)$.
The smallest such value of p is the period of f.

* Periods of trigonometric functions:

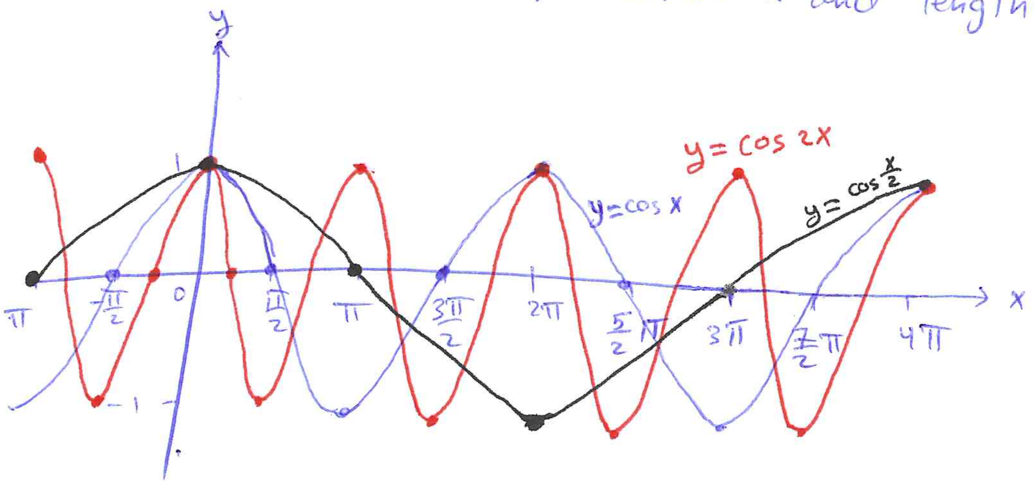
See page 29

Period π examples: $\tan x = \tan(x+\pi)$
$\cot x = \cot(x+\pi)$

Period 2π examples: $\sin x = \sin(x+2\pi)$
$\cos x = \cos(x+2\pi)$
$\sec x = \sec(x+2\pi)$
$\csc x = \csc(x+2\pi)$

Example: Draw ① $y = \cos 2x$
② $y = \cos \frac{x}{2}$

Note that ① Multiplying x by a number greater than 1 speeds up the trigonometric function (increase the frequency). ($p\downarrow$)
② Multiplying x by a positive number less than 1 slows the trigonometric function down and lengthens its period ($p\uparrow$).



$\cos x$ has $p = 2\pi$
$\cos \frac{x}{2}$ has $p = 4\pi$
$\cos 2x$ has $p = \pi$

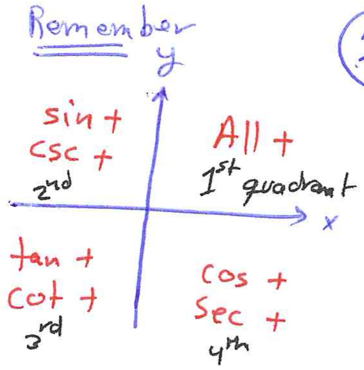
* Even trigonometric functions:

$$\cos(-x) = \cos x$$
$$\sec(-x) = \sec x$$

* Odd trigonometric functions:

$$\sin(-x) = -\sin x$$
$$\tan(-x) = -\tan x$$
$$\csc(-x) = -\csc x$$
$$\cot(-x) = -\cot x$$

Remember



see also page 29

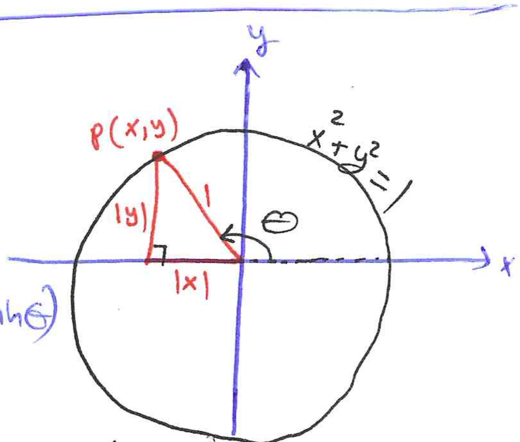
* Identities

Recall the unit cycle $x^2+y^2=1$

Remembe that the point $p(x,y) = p(\cos\theta, \sin\theta)$

Apply Pythagorean theorem $\Rightarrow$

$$|x|^2+|y|^2 = 1^2 \Rightarrow \boxed{\cos^2\theta + \sin^2\theta = 1} \quad (1)$$



The triangle for a general angle θ

* Divid equation (1) by $\cos^2\theta$, we get

$$\boxed{1 + \tan^2\theta = \sec^2\theta}$$

- Divid equation (1) by $\sin^2\theta$, we get

$$\boxed{1 + \cot^2\theta = \csc^2\theta}$$

Angle Sum Formulas:

$$\boxed{\begin{aligned} \cos(A+B) &= \cos A \cos B - \sin A \sin B \\ \sin(A+B) &= \sin A \cos B + \cos A \sin B \end{aligned}} \quad (2)$$

* Double-angle Formulas:

Make $A = B = \theta$ in equation (2), we get

$$\boxed{\begin{array}{l} \cos 2\theta = \cos^2\theta - \sin^2\theta \\ \hline \sin 2\theta = 2\sin\theta\cos\theta \end{array}} \quad (3)$$

* Additional double-angle Formulas:

→ Add equation (1) to equation (3), we get

$$2\cos^2\theta = 1 + \cos 2\theta \Rightarrow \boxed{\cos^2\theta = \frac{1+\cos 2\theta}{2}} \quad (4)$$

→ substract (3) from (1), we get

$$(1) - (3) \Rightarrow 2\sin^2\theta = 1 - \cos 2\theta \Rightarrow \boxed{\sin^2\theta = \frac{1-\cos 2\theta}{2}} \quad (5)$$

* Half-angles formulas:

→ Apply $\frac{\theta}{2}$ in equation (4), we get

$$\boxed{\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}}$$

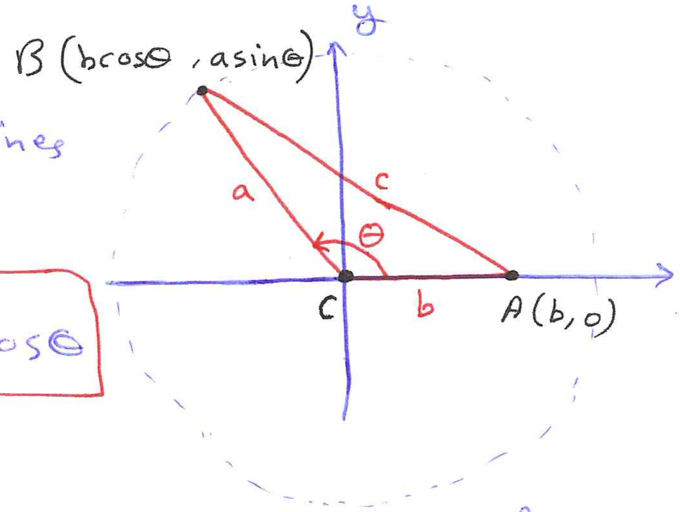
→ Apply $\frac{\theta}{2}$ in equation (5), we get

$$\boxed{\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}}$$

The law of Cosines

Consider the triangle ABC. The law of Cosines is given by

$$\boxed{c^2 = a^2 + b^2 - 2ab\cos\theta}$$



Proof: The distance between A and B is given by

$$c^2 = (a\cos\theta - b)^2 + (a\sin\theta)^2$$

$$= a^2\cos\theta^2 - 2ab\cos\theta + b^2 + a^2\sin\theta^2$$

$$= a^2(\cos\theta^2 + \sin\theta^2) + b^2 - 2ab\cos\theta$$

$$= a^2 + b^2 - 2ab\cos\theta$$

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* Note that the law of cosines generalizes the Pythagorean theorem. If $\theta = \frac{\pi}{2}$, then $\cos\frac{\pi}{2} = 0$ and $c^2 = a^2 + b^2$.

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* Transformation of Trigonometric functions:

$$y = a\,f(b(x+c)) + d$$

vertical stretch or compression
reflection about x-axis if $a<0$

vertical shift

Horizontal shift

• Horizontal stretch or compression
• reflection about y-axis if $b<0$

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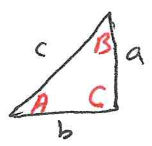
* For any angle θ measured in radians:

$$-|\theta| \le \sin\theta \le |\theta| \quad \text{and} \quad -|\theta| \le 1-\cos\theta \le |\theta|$$

Note that the law of sines says that if a, b, c are sides opposites the angles A, B, C in a triangle, then



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$