Chapter 1

2.1 | 2.2 Limits and Continuity

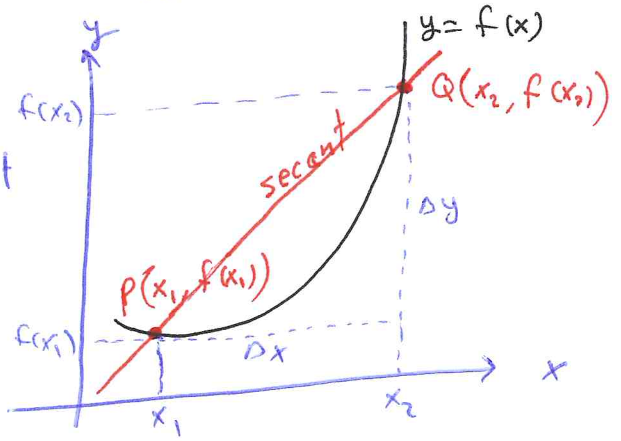
Rates of change and limits

Def: The average rate of change of the function $y=f(x)$ w.r.t x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{f(x_1+h)-f(x_1)}{h}$$

h is the length of the interval

* Note that the average rate of change = slope of the secant



Example 1: Find the average rate of change of the function $f(x)=\sqrt{x}$ over $[4,9]$

The average rate of change $= \frac{\Delta y}{\Delta x} = \frac{f(9)-f(4)}{9-4} = \frac{\sqrt{9}-\sqrt{4}}{9-4} = \frac{3-2}{5} = \frac{1}{5}$

Example 2 : see the end

2.2 * limits of function values

Example: How does the function $f(x)=\frac{x^2-1}{x-1}$ behave near $x=1$?

$\Rightarrow$ The problem is when $x=1$ (we can't divid over zero)

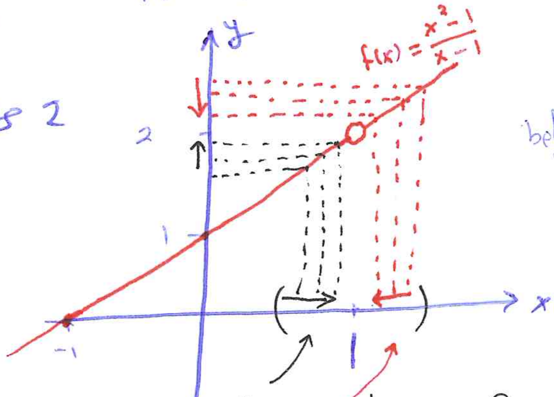
$f(x) = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1$ for all values of x except $x=1$

$\Rightarrow$ We say $f(x)$ approches 2 as x approches 1.

We write this as

$\lim_{x\to 1} f(x) = 2$ or

$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$



below 1

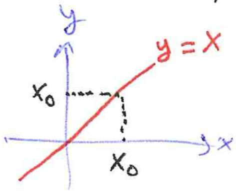
x	f(x)
0.9	1.9
0.99	1.99
0.999	1.999
1	
1.001	2.001
1.01	2.01
1.1	2.1

x approches 1 from left $x\to 1^-$

x approches 1 from right $x\to 1^+$

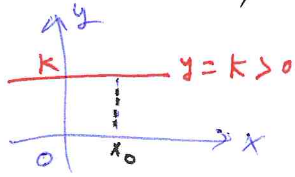
Example: [a] If f is the identity function $f(x) = x$, then for any value x_0

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0$$

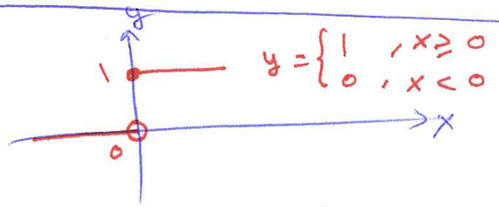


[b] If f is the constant function $f(x) = k$, then for any value x_0

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} k = k$$



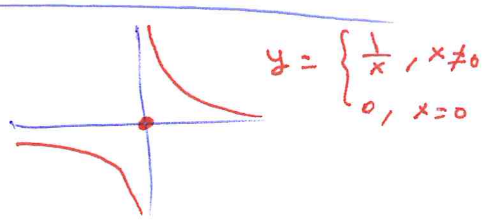
Example:



$\lim_{x \to 0} f(x)$ DNE because at $x = 0$, y jumbs

As $x \to 0^-$, $y \to 0$

As $x \to 0^+$, $y \to 1$



$\lim_{x \to 0} f(x)$ DNE

As $x \to 0^-$, $y \to -\infty$

As $x \to 0^+$, $y \to \infty$

Theorem 1 (limit laws) If L, M, c, k are real numbers and $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$

then:

1. Sum Rule: $\lim_{x \to c} (f(x) + g(x)) = L + M$
2. Difference Rule: $\lim_{x \to c} (f(x) - g(x)) = L - M$
3. Constant Multiply Rule: $\lim_{x \to c} k f(x) = kL$
4. Product Rule: $\lim_{x \to c} f(x) g(x) = LM$
5. Quotient Rule: $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$, $M \neq 0$
6. Power Rule: $\lim_{x \to c} (f(x))^n = L^n$, n is positive integer
7. Root Rule: $\lim \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{\frac{1}{n}}$, $n = =$

If n is even, we assume that $L > 0$.

Example: Find

(a) $\lim_{x\to 1}(x^3-4x^2+1) = (1)^3-4(1)^2+1 = 1-4+1 = -2$

(b) $\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = 2+2 = 4$

(c) $\lim_{x\to -2}\sqrt{4x^2-7} = \lim \sqrt{4(-2)^2-7} = \sqrt{16-7} = \sqrt{9} = 3$

Theorem 2 (limits of Polynomials)

If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then

$$\lim_{x\to c} p(x) = p(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_1 c + a_0$$

Theorem 3 (limits of Rational functions)

If $p(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$,

then $\lim_{x\to c}\frac{p(x)}{Q(x)} = \frac{p(c)}{Q(c)}$

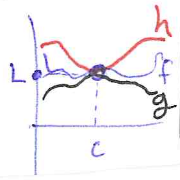
Example: Find $\lim_{x\to -1}\frac{x^3+2x^2-1}{x^2+3} = \frac{(-1)^3+2(-1)^2-1}{(-1)^2+3} = \frac{-1+2-1}{4} = \frac{0}{4} = 0$

Example: Find $\lim_{x\to 1}\frac{x^2+x-2}{x^2-x} = \lim_{x\to 1}\frac{\cancel{(x-1)}(x+2)}{x\cancel{(x-1)}} = \lim_{x\to 1}\frac{x+2}{x} = \frac{3}{1} = 3$

Example: Find $\lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2} = \lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2}\cdot\frac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3}$ multiply by the conjugate of numerator

$$= \lim_{x\to 0}\frac{\cancel{x^2}+\cancel{9-9}}{\cancel{x^2}(\sqrt{x^2+9}+3)} = \lim_{x\to 0}\frac{1}{\sqrt{x^2+9}+3} = \frac{1}{3+3} = \frac{1}{6}$$

Theorem 4 (Sandwich theorem) Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$. Suppose also that $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$. Then $\lim_{x\to c} f(x) = L$



Example: Given that $1-\frac{x^2}{4} \le u(x) \le 1+\frac{x^2}{2}$ for all $x \neq 0$
find $\lim\limits_{x\to 0} u(x)$.

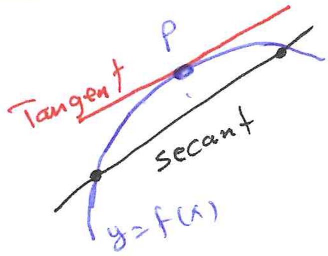
$$\lim_{x\to 0}\left(1-\frac{x^2}{4}\right) = 1 = \lim_{x\to 0}\left(1+\frac{x^2}{2}\right)$$

Thus, by sandwich theorem $\lim\limits_{x\to 0} u(x) = 1$

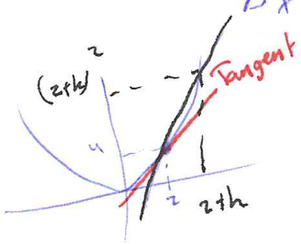
Theorem 5: If $f(x) \le g(x)$ $\forall x$ in some open interval containing c, except possibly at $x=c$, and the limits of f and g both exist as x approaches c, then

$$\lim_{x\to c} f(x) \le \lim_{x\to c} g(x)$$

Example: Find the slope of $y = x^2$ at point $(2,4)$.
Write an equation for the tangent at this point.



$$\text{Secant Slope} = \frac{\Delta y}{\Delta x} = \frac{f(x_1+h)-f(x_1)}{h} = \frac{f(2+h)-f(2)}{h} = \frac{(2+h)^2-2^2}{h}$$



$$= \frac{\cancel{4}+4h+h^2-\cancel{4}}{h} = \frac{h(4+h)}{h} = 4+h$$

$$\text{Tangent slope} = \lim_{h\to 0} \text{Secant slop} = 4$$

$$y - y_0 = m(x - x_0)$$

$$y - 4 = 4(x-2) = 2x - 8$$

$$y = 4x - 4$$