The Precise Definitions of Limits

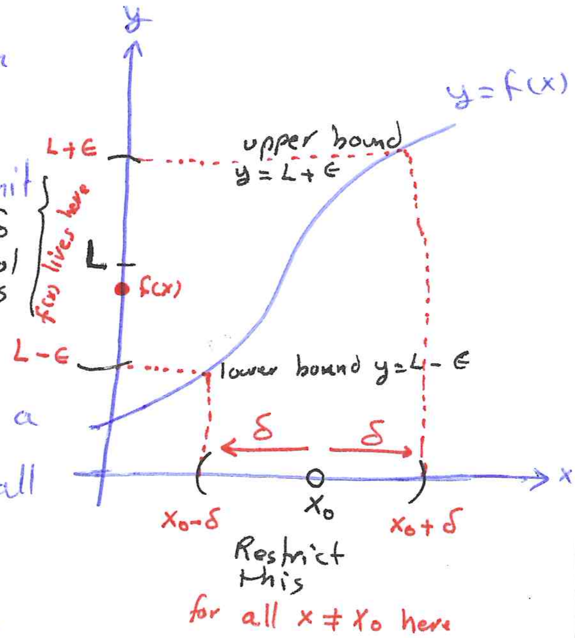
Def: Example $f(x) = 2x+1$, $\epsilon = 2 \Rightarrow \delta = 1$

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say $f(x)$ approaches the limit L as x approaches x_0, and we write:

$$\lim_{x \to x_0} f(x) = L,$$

if for every number $\epsilon > 0$, there exists a corr sponding number $\delta > 0$ such that for all x

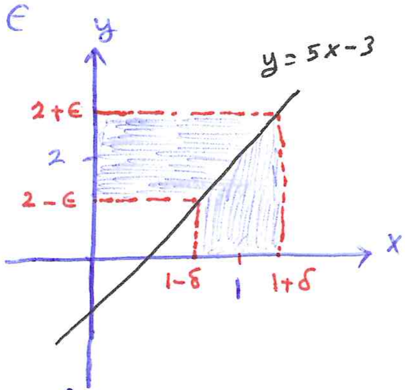
$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon.$$



Example: show that $\lim_{x \to 1} (5x - 3) = 2$

→ Let $\epsilon > 0$, we must find $\delta > 0$ s.t for all x
if $|x - x_0| < \delta$ then $|f(x) - L| < \epsilon$

→ $x_0 = 1$, $L = 2$, $f(x) = 5x - 3$
That is we need to find $\delta > 0$ s.t for all x if $|x - 1| < \delta$ then
$|(5x-3) - 2| < \epsilon$



$\Rightarrow |5x - 5| < \epsilon \Rightarrow 5|x - 1| < \epsilon \Rightarrow |x - 1| < \frac{\epsilon}{5}$

Thus, we can take $0 < \delta \le \frac{\epsilon}{5}$ because if

$|x - 1| < \delta = \frac{\epsilon}{5}$, then $|(5x-3) - 2| = |5x - 5| = 5|x - 1| < \cancel{5}\frac{\epsilon}{\cancel{5}} = \epsilon$

To find the maximum δ^* in which $0 < \delta \le \delta^*$, we may use $f(x_0 - \delta^*) = L - \epsilon$ or $f(x_0 + \delta^*) = L + \epsilon$

In the previous example:

$f(x_0 - \delta^*) = L - \epsilon \Rightarrow f(1 - \delta^*) = 2 - \epsilon \Rightarrow$

$5(1 - \delta^*) - 3 = 2 - \epsilon \Rightarrow 5 - 5\delta^* - 3 = 2 - \epsilon \Rightarrow \cancel{2} - 5\delta^* = \cancel{2} - \epsilon$

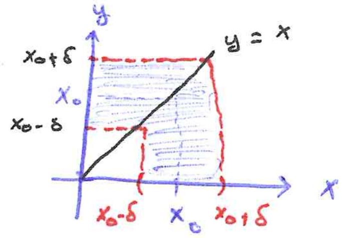
$\Rightarrow \boxed{\delta^* = \frac{\epsilon}{5}}$

Example: show that $\lim_{x \to x_0} x = x_0$ $\quad f(x) = x$ and $L = x_0$

Let $\epsilon > 0$, we need to find $\delta > 0$ such that for all x

if $|x - x_0| < \delta$ then $|x - x_0| < \epsilon$

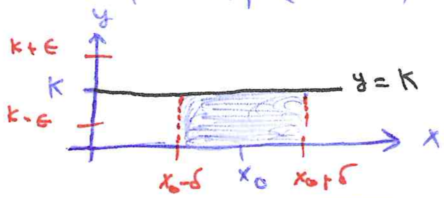
Take $0 < \delta \le \epsilon$. This proves that $\lim_{x \to x_0} x = x_0$



Example: Prove that $\lim_{x \to x_0} k = k$

Let $\epsilon > 0$, we need to find $\delta > 0$ s.t for all x

if $|x - x_0| < \delta$ then $|k - k| < \epsilon$.



$0 < \epsilon$ This is true for any $\delta > 0$.

This proves that $\lim_{x \to x_0} k = k$

How to find δ for a given f, L, x_0 and ϵ:

Two steps to find $\delta > 0$ s.t for all x

if $|x - x_0| < \delta$ then $|f(x) - L| < \epsilon$

$\boxed{1}$ Solve the inequality $|f(x) - L| < \epsilon$ to find an open interval (a, b) about x_0 on which the inequality holds for all $x \neq x_0$.

$\boxed{2}$ Find $\delta > 0$ that places the open interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 inside the interval (a, b). The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq x_0$ in this δ-interval.

Example: Prove that $\lim_{x \to 2} f(x) = 4$ where $f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2 \end{cases}$

Let $\epsilon > 0$, we need to find $\delta > 0$ s.t for all x

if $|x - 2| < \delta$ then $|f(x) - 4| < \epsilon$

step $\boxed{1}$: solve $|f(x) - 4| < \epsilon$ to find an open interval about $x_0 = 2$ in which the inequality $|f(x) - 4| < \epsilon$ holds for all $x \neq x_0$

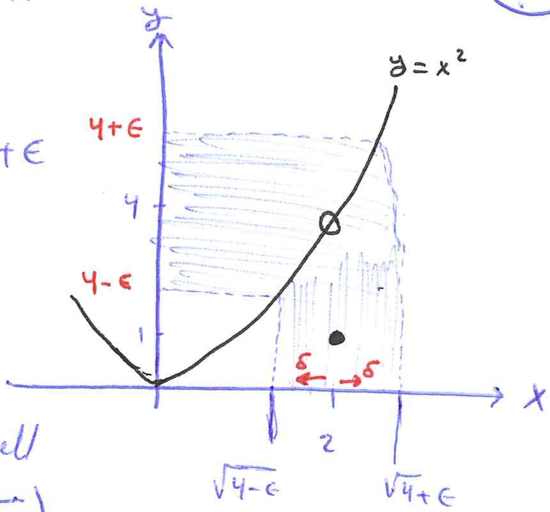
$\Rightarrow$ For $x \neq x_0 = 2$ we have $f(x) = x^2$

$\Rightarrow |x^2 - 4| < \epsilon \Rightarrow$

$-\epsilon < x^2 - 4 < \epsilon \Rightarrow 4 - \epsilon < x^2 < 4 + \epsilon$

$\Rightarrow \sqrt{4-\epsilon} < |x| < \sqrt{4+\epsilon}$ (Assume $\epsilon < 4$)

$\sqrt{4-\epsilon} < x < \sqrt{4+\epsilon}$



The inequality $|f(x) - 4| < \epsilon$ holds for all $x \neq 2$ in the open interval $(\sqrt{4-\epsilon}, \sqrt{4+\epsilon})$.

step $\boxed{2}$ Find $\delta > 0$ that places the centered interval $(2-\delta, 2+\delta)$ inside the interval $(\sqrt{4-\epsilon}, \sqrt{4+\epsilon})$

Take $\delta = \min\{2 - \sqrt{4-\epsilon}, \sqrt{4+\epsilon} - 2\}$.

Thus, the inequality $|x-2| < \delta$ will automatically place x between $\sqrt{4-\epsilon}$ and $\sqrt{4+\epsilon}$ to make $|f(x) - 4| < \epsilon$.

Example: Given that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. We now can prove Theorems

Prove that $\lim_{x \to c} (f(x) + g(x)) = L + M$.

Let $\epsilon > 0$, we need to find $\delta > 0$ s.t for all x

if $|x - c| < \delta$ then $|f(x) + g(x) - (L+M)| < \epsilon$.

> since $\lim_{x \to c} f(x) = L$, $\exists\, \delta_1 > 0$ s.t for all x if $|x-c| < \delta_1$ then $|f(x) - L| < \frac{\epsilon}{2}$

> Since $\lim_{x \to c} g(x) = M$, $\exists\, \delta_2 > 0$ s.t for all x if $|x - c| < \delta_2$, then $|g(x) - M| < \frac{\epsilon}{2}$

step $\boxed{1}$ $|f(x) + g(x) - (L+M)| = |(f(x) - L) + (g(x) - M)|$ Triangle Inequality

$\leq |f(x) - L| + |g(x) - M|$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

step $\boxed{2}$ Take $\delta = \min\{\delta_1, \delta_2\}$, so that

If $|x - c| < \delta_1$ then $|x-c| < \delta$, thus $|f(x) - L| < \frac{\epsilon}{2}$.

If $|x - c| < \delta_2$ then $|x - c| < \delta$, thus $|g(x) - M| < \frac{\epsilon}{2}$. Thus