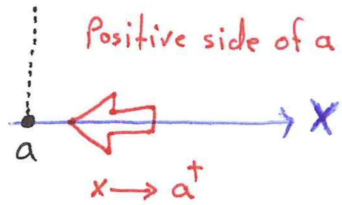
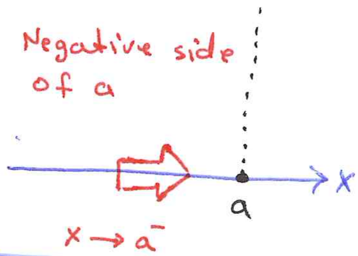
A: One-sided limits: is two parts:

1. **Right-hand limit** $(x \to a^+)$ means x approaches a from the positive side of a through values greater than a.

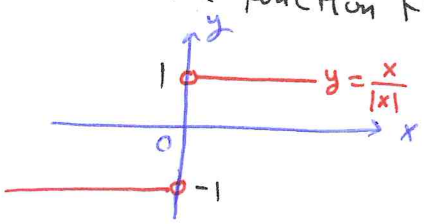


2. **Left-hand limit** $(x \to a^-)$ means x approaches a from the negative side of a through values less than a.



Example 1 Find $\lim\limits_{x \to 0^+} f(x)$ and $\lim\limits_{x \to 0^-} f(x)$ for the function $f(x) = \dfrac{x}{|x|}$

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$



$\lim\limits_{x \to 0^-} f(x) = -1$ (left-hand limit) and $\lim\limits_{x \to 0^+} f(x) = 1$ (Right-hand limit)

Let $f(x)$ be defined on an interval (a, b) where $a < b$. If $f(x)$ approaches L as x approaches a within the interval (a, b), then we say f has **right-hand limit** L at a, and we write

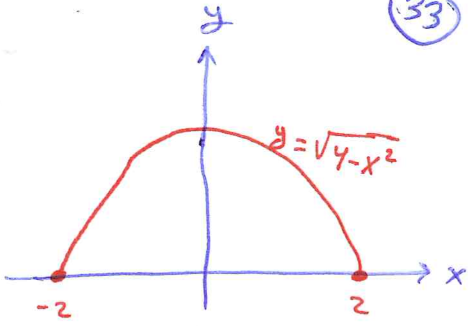
$$\lim_{x \to a^+} f(x) = L$$

Let $f(x)$ be defined on an interval (c, a) where $c < a$. If $f(x)$ approaches M as x approaches a within the interval (c, a), then we say f has **left-hand limit** M at a, and we write

$$\lim_{x \to a^-} f(x) = M$$

Example $\boxed{2}$ Let $f(x) = \sqrt{4-x^2}$. Find

(a) $\lim_{x \to 2^-} f(x)$ (b) $\lim_{x \to -2^+} f(x)$



Note that the domain of $f(x)$ i $[-2,2]$ and the graph of f is semicircle

(a) $\lim_{x \to 2^-} f(x) = 0$ (b) $\lim_{x \to -2^+} f(x) = 0$

(c) $\lim_{x \to -2^-} f(x) = DNE$ (d) $\lim_{x \to 2^+} f(x) = DNE$

Theorem (~~One-sided vs. Two-sided Limits~~)

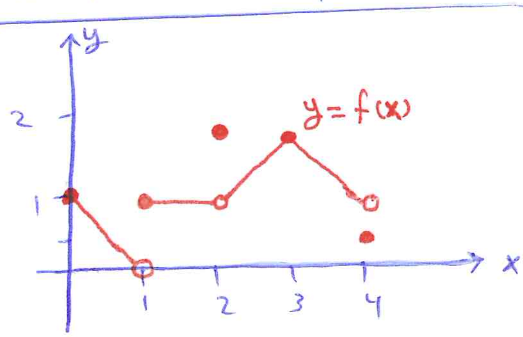
A function $f(x)$ has a limit L as x approaches c iff the one-sided limits of f are equal:

$$\lim_{x \to c} f(x) = L \iff \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$$

Note that in Example 1: $\lim_{x \to 0} f(x)$ does not exist because the right and left hand limits are not equal.

Example 2: $\lim_{x \to 2} f(x)$ and $\lim_{x \to -2} f(x)$ do not exist because the right and left hand limits are not equal.

Example $\boxed{3}$ Consider the graph $y = f(x)$. Find



(a) $\lim_{x \to 0^+} f(x) = 1$

$\lim_{x \to 0^-} f(x) = DNE$

$\lim_{x \to 0} f(x) = DNE$

(b) $\lim_{x \to 1^-} f(x) = 0$ $f(1) = 1$

$\lim_{x \to 1^+} f(x) = 1$

$\lim_{x \to 1} f(x) = DNE$

(c) $\lim_{x \to 2^-} f(x) = 1$ $f(2) = 2$

$\lim_{x \to 2^+} f(x) = 1$

$\lim_{x \to 2} f(x) = 1$

(d) $\lim_{x \to 3^-} f(x) = 2$ $f(3) = 2$

$\lim_{x \to 3^+} f(x) = 2$

$\lim_{x \to 3} f(x) = 2$

(e) $\lim_{x \to 4^-} f(x) = 1$ $f(4) = \frac{1}{2}$

$\lim_{x \to 4^+} f(x) = DNE$

$\lim_{x \to 4} f(x) = DNE$

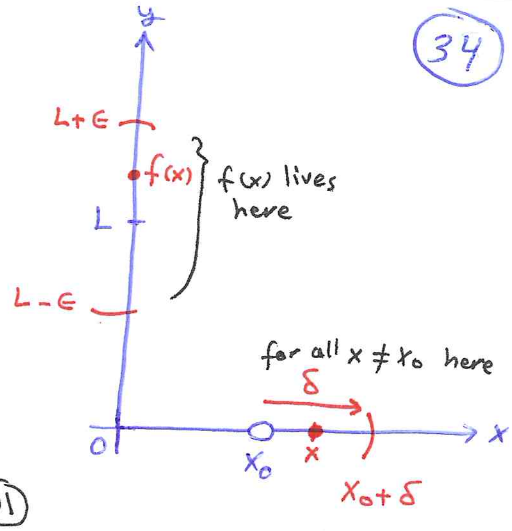
Definition (Right-hand limit)

$f(x)$ has right-hand limit L at x_0

$$\lim_{x \to x_0^+} f(x) = L$$

if for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x

$$\boxed{\text{if } 0 < x - x_0 < \delta \text{ then } |f(x) - L| < \epsilon} \to \text{D1}$$



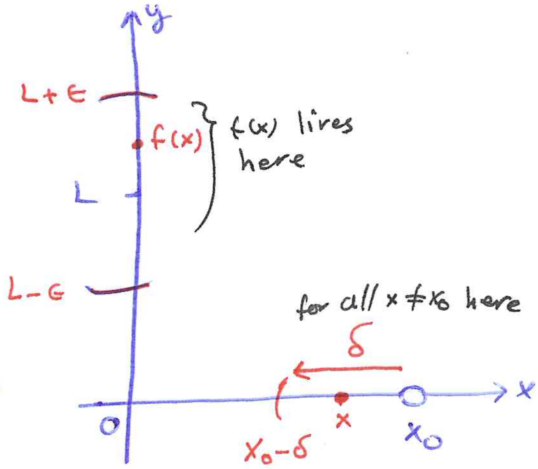
Definition (Left-hand limit)

$f(x)$ has left-hand limit L at x_0

$$\lim_{x \to x_0^-} f(x) = L$$

if for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x

$$\boxed{\text{if } -\delta < x - x_0 < 0 \text{ then } |f(x) - L| < \epsilon} \to \text{D2}$$



* Note that the relation between One and Two sided Limits is clear from D1 and D2:

→ For the right-hand limit we have D1

→ For the left-hand limit we have D2

$\boxed{\text{D1}}$ together with $\boxed{\text{D2}}$ provid the two sided limit

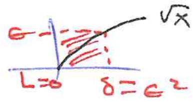
$$\text{if } \quad |x - x_0| < \delta \quad \text{then} \quad |f(x) - L| < \epsilon$$

Thus, $\displaystyle\lim_{x \to x_0} f(x) = L$

Example: Prove that $\displaystyle\lim_{x \to 0^+} \sqrt{x} = 0$ $\qquad L = 0$, $x_0 = 0$, $f(x) = \sqrt{x}$

let $\epsilon > 0$, we must find $\delta > 0$ s.t for all x if $0 < x - 0 < \delta$ then $|\sqrt{x} - 0| < \epsilon$

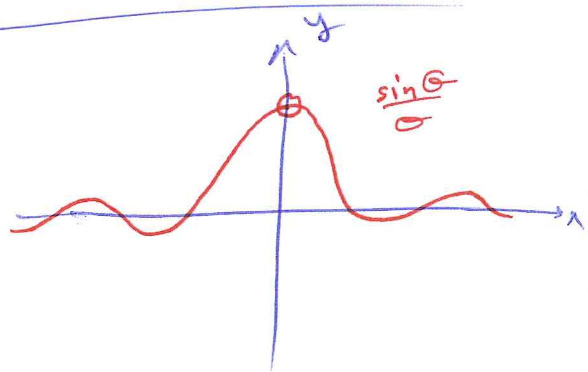
$0 < x < \delta$ then $\sqrt{x} < \epsilon$ $\quad x < \epsilon^2$ $\Rightarrow$ $0 < \delta < \epsilon^2$



Theorem 7: $\displaystyle\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ (θ in radians)

Example: Show that $\displaystyle\lim_{x \to 0} \frac{\sin 3x}{4x} = \frac{3}{4}$

$$\lim_{x \to 0} \frac{\sin 3x}{4x} = \lim_{x \to 0} \frac{\sin 3x}{\frac{4}{3}\, 3x} = \frac{3}{4} \lim_{x \to 0} \frac{\sin 3x}{3x} = \frac{3}{4}$$



Example: Find $\displaystyle\lim_{t \to 0} \frac{\tan t \sec 2t}{3t}$

$$\lim_{t \to 0} \frac{\sin t}{\cos t} \frac{1}{3t} \frac{1}{\cos 2t} = \lim_{t \to 0} \frac{1}{3} \frac{\sin t}{t} \frac{1}{\cos t} \frac{1}{\cos 2t}$$

$$= \frac{1}{3}(1)(1)(1) = \frac{1}{3}$$
