3.3 Differentiation Rules

① Derivative of a constant function

If $f(x) = c$, then $\frac{df}{dx} = 0$

Proof: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0$

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② Power Rule

If $f(x) = x^n$, then $f'(x) = n x^{n-1}$ $\quad n \in \mathbb{R}$

for all x, where x^n and x^{n-1} are defined.

Example ① Let $f(x) = \sqrt{3}$. $\Rightarrow f'(x) = 0$

② $f(x) = x^3 \Rightarrow f'(x) = 3x^2$

③ $f(x) = \frac{1}{x^4} \Rightarrow f'(x) = -4x^{-5} = \frac{-4}{x^5}$

④ $g(x) = \sqrt{x^{2+\pi}} \Rightarrow g(x) = \left(x^{2+\pi}\right)^{\frac{1}{2}} = x^{1+\frac{\pi}{2}}$

$\Rightarrow g'(x) = \left(1 + \frac{\pi}{2}\right) x^{\frac{\pi}{2}} = \left(1 + \frac{\pi}{2}\right)\sqrt{x^{\pi}}$

⑤ $h(x) = x^{\sqrt{3}} \Rightarrow h'(x) = \sqrt{3}\, x^{\sqrt{3}-1}$

Let $f(x) = c\,u(x)$

③ where $u(x)$ is a differentiable function of x and c is a constant, then

$$f'(x) = \frac{d}{dx}(cu) = c\frac{du}{dx} = c\,u'(x)$$

Proof: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \dfrac{c\,u(x+h) - c\,u(x)}{h}$

$$= c \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} = c\,u'(x)$$

Example: ① If $f(x) = 5x^2$, then $f'(x) = (5)(2)x = 10x$

② If $f(x) = -\sqrt{3}\,x^3$, then $f'(x) = -3\sqrt{3}\,x^2$

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④ Derivative sum Rule:

If $f(x) = u(x) + v(x)$ where u and v are differentiable functions of x, then f is differentiable at every point where u and v are both differentiable

$$f'(x) = u'(x) + v'(x)$$

Proof: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \dfrac{u(x+h) + v(x+h) - u(x) - v(x)}{h}$

$$= \lim_{h \to 0} \left[\frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h} \right]$$

$$= u'(x) + v'(x)$$

Example ① Let $f(x) = x^3 - \frac{3}{2}x^2 - 7x + 3$.

$f'(x) = 3x^2 - 3x - 7$

② Does the curve $g(x) = x^2 + 1$ have any horizontal tangents?

Horizontal tangents $\Rightarrow g'(x) = 0 = 2x \Rightarrow \boxed{x = 0}$

$(0, g(0)) = (0, 1)$

5 Derivative Product Rule:

Let $f(x) = g(x)\, p(x)$, where g and p are differentiable at x, then f is differentiable at x:

$$f'(x) = g(x)\, p'(x) + p(x)\, g'(x)$$

Example: Find the derivative of $y = (x^2-1)(x^3+3)$

$$y' = (x^2-1)(3x^2) + (x^3+3)(2x)$$
$$= 3x^4 - 3x^2 + 2x^4 + 6x$$
$$= 6x^4 - 3x^2 + 6x$$

Proof: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{g(x+h)p(x+h) - g(x)p(x)}{h}$

$$= \lim_{h\to 0} \frac{g(x+h)p(x+h) - g(x+h)p(x) + g(x+h)p(x) - g(x)p(x)}{h}$$

$$= \lim_{h\to 0} \left[g(x+h) \frac{p(x+h)-p(x)}{h} + p(x) \frac{g(x+h)-g(x)}{h} \right]$$

$$= \lim_{h\to 0} g(x+h) \lim_{h\to 0} \frac{p(x+h)-p(x)}{h} + p(x) \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= g(x)\, p'(x) + p(x)\, g'(x)$$

6 Derivative Quotient Rule:

Let $f(x) = \frac{g(x)}{p(x)}$, $p(x) \neq 0$, p and g are differentiable

then f is differentiable at x:

$$f'(x) = \frac{p(x)\, g'(x) - g(x)\, p'(x)}{p^2(x)}$$

Example: Find the derivative of $y = \dfrac{3x-4}{x^2+1}$

$$y' = \frac{(x^2+1)(3) - (3x-4)(2x)}{(x^2+1)^2}$$

$$= \frac{3x^2+3 - 6x^2+8x}{(x^2+1)^2} = \frac{3+8x-3x^2}{(x^2+1)^2}$$

Proof: $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{g(x+h)}{p(x+h)} - \dfrac{g(x)}{p(x)}}{h}$$

$$= \lim_{h\to 0} \frac{p(x)\,g(x+h) - g(x)\,p(x+h)}{h\,p(x)\,p(x+h)}$$

$$= \lim_{h\to 0} \frac{p(x)\,g(x+h) - p(x)\,g(x) + p(x)\,g(x) - g(x)\,p(x+h)}{h\,p(x)\,p(x+h)}$$

$$= \lim_{h\to 0} \frac{p(x)\dfrac{g(x+h)-g(x)}{h} - g(x)\dfrac{p(x+h)-p(x)}{h}}{p(x)\,p(x+h)}$$

$$= \frac{p(x)\,g'(x) - g(x)\,p'(x)}{p^2(x)}$$

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6 Second and Higher-Order Derivatives:

Let $y = f(x)$

$y' = f'(x) = \dfrac{dy}{dx}$ 1st derivative "y prime"

$y'' = f''(x) = \dfrac{d^2y}{dx^2}$ 2nd derivative "y double prime"

$y''' = f'''(x) = \dfrac{d^3y}{dx^3}$ 3rd derivative "y trible prime"

$\vdots$

$y^{(n)} = f^{(n)}(x) = \dfrac{d^ny}{dx^n}$ nth derivative "y super n"

Assuming f has all derivative

n is positive integer

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^2\, f(x).$$

Example: Find the derivatives of all orders of the function $y = x^3 - 3x^2 + 2$

First derivative $y' = 3x^2 - 6x$

Second derivative $y'' = 6x$

Third derivative $y''' = 6$

Fourth derivative $y^{(4)} = 0$

The fifth and later derivatives are zero.

Proof of 2 $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{z \to x} \dfrac{f(z) - f(x)}{z - x}$

$$= \lim_{z \to x} \frac{z^n - x^n}{z - x}$$

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$$= \lim_{z \to x} \frac{(z-x)(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1})}{(z-x)}$$

$$= \lim_{z \to x} (z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1})$$

$$= n\,x^{n-1}$$
