

3.4 The Derivative as a Rate of Change (58)

Definition: The instantaneous rate of change of  $f$  w.r.t  $x$  at  $x_0$  is

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} \quad \text{"limit exists"}$$

Example: Let  $A = \frac{\pi}{4} D^2$  A: area of a circle  
D: diameter of circle

Find the instantaneous rate of change of the area w.r.t the diameter when  $D = 10$  m

$$\left. \frac{dA}{dD} \right|_{D=10} = \left. \frac{\pi}{4} 2D \right|_{D=10} = \left. \frac{\pi}{2} D \right|_{D=10} = \frac{\pi}{2} (10) = 5\pi \text{ m}^2/\text{m}$$

Suppose an object is moving accordingly to

$t$ : time  
 $s$ : distance

$s = f(t)$

• The displacement of the object over the time interval  $[t_1, t_2]$  is  $\Delta t = t_2 - t_1$

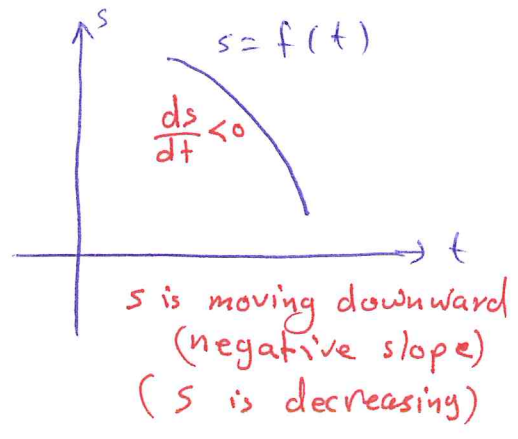
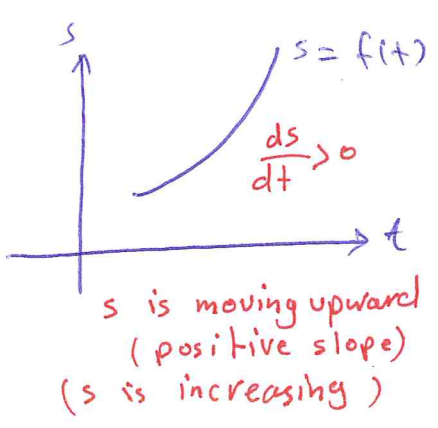
$$\Delta s = f(t_2) - f(t_1) = f(t_1 + \Delta t) - f(t_1)$$

• The average velocity of the object over the time interval  $[t_1, t_2]$  is

$$V_{av} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t_1 + \Delta t) - f(t_1)}{\Delta t}$$

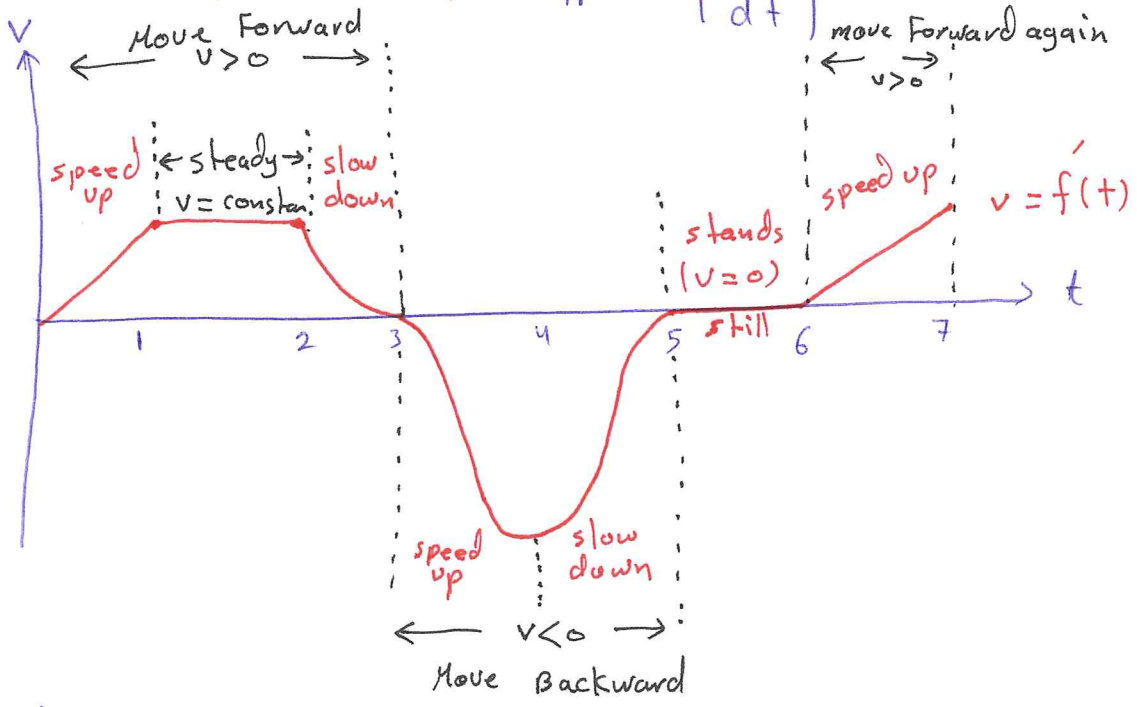
The velocity (instantaneous velocity) at time  $t$  is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}$$



Speed is the absolute value of velocity

$$\text{Speed} = |v(t)| = \left| \frac{ds}{dt} \right|$$



Acceleration is the derivative of velocity w.r.t time.

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Jerk is the derivative of acceleration w.r.t. time

$$j(t) = \frac{da}{dt} = \frac{d^2v}{dt^2} = \frac{d^3s}{dt^3}$$

Example The free falling equation for an object falls from a rest is  $s = \frac{1}{2} g t^2$

where  $g$  is the gravity acceleration =  $9.8 \text{ m/s}^2$

$$s = \frac{1}{2} (9.8) t^2 = 4.9 t^2$$

(a) How many meters does the object fall in the first 2s

$$s(2) = 4.9 (2)^2 = 4.9 (4) = 19.6 \text{ m}$$

(b) What is its velocity at  $t=2$ ?

$$v(t) = s'(t) = 2 (4.9) t = 9.8 t$$

$$v(2) = 9.8 (2) = 19.6 \text{ m/s}$$

(c) what is its speed at  $t=2$ ?

$$\text{speed} \Big|_{t=2} = |v(2)| = 19.6 \text{ m/s}$$

(d) what is its acceleration at  $t=2$ ?

$$a(t) = \frac{dv}{dt} = 9.8$$

$$a(2) = 9.8 \text{ m/s}^2$$

(e) what is its jerk at  $t=2$

$$j(t) = a'(t) = 0 \quad \text{Thus } j(2) = 0 \text{ m/s}^3$$

Example: A rock straight up with a launch velocity of  $50 \text{ m/s}$ . It reaches a height of  $s = 49t - 4.9t^2 \text{ m}$  after  $t$  second.

(a) How high does the rock go?

velocity = 0

$$v(t) = s'(t) = 49 - 9.8t = 0 \Leftrightarrow t = 5 \text{ second}$$

$$\Rightarrow \text{The rock's height at } t=5 \text{ is } s(5) = 49(5) - 4.9(5)^2 = 122.5 \text{ m}$$

[b] what is the velocity and speed when the rock is 78.4 m above the ground on the way above and on the way down? (61)

$$s(t) = 49t - 4.9t^2 = 78.4$$

$$4.9t^2 - 49t + 78.4 = 0$$

$$4.9(t^2 - 10t + 16) = 0$$

$$(t-2)(t-8) = 0$$

$$t = 2 \text{ seconds}$$

$$t = 8 \text{ seconds}$$

$$v(t) = 49 - 9.8t$$

$$v(2) = 49 - 9.8(2) = 29.4 \text{ m/s} \quad \text{"The rock is moving up"}$$

$$v(8) = 49 - 9.8(8) = -29.4 \text{ m/s} \quad \text{" = = = = down"}$$

The speed is  $|v(t)| = 29.4 \text{ m/s}$

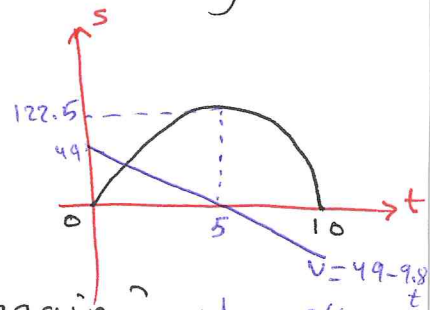
[c] what is the acceleration of the rock at any time?

$$a(t) = \dot{v}(t) = -9.8 \text{ m/s}^2$$

The acceleration is always downward.

As the rock rises, it slows down.

As the rock falls, it speeds up.



[d] When does the rock hit the ground again? when  $s(t) = 0$

$$s(t) = 49t - 4.9t^2 = 0$$

$$4.9t(10-t) = 0 \Leftrightarrow t = 0 \text{ sec}, t = 10 \text{ sec.}$$

### Derivative in Economics:

•  $c(x)$ : cost of production "to produce  $x$ "

• Marginal cost of production:  $c'(x) = \lim_{h \rightarrow 0} \frac{c(x+h) - c(x)}{h}$

•  $f'(x)$  measures the sensitivity  
"small change in  $x$  produce large change in the value of a function  $f(x)$ "

