$\boxed{3.5}$ Derivatives of Trigonometric Functions

$\boxed{1}$ If $f(x) = \sin x$, then $f'(x) = \cos x$

Proof:
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x (\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h\to 0} \left(\sin x \frac{\cos h - 1}{h} \right) + \lim_{h\to 0} \left(\cos x \frac{\sin h}{h} \right)$$

$$= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h}$$

$$= \sin x \, (0) + \cos x \, (1) = \cos x$$

Example: Find y' for the following functions:

$\boxed{a}$ $y = 3x - \sin x$ $\qquad y' = 3 - \cos x$

$\boxed{b}$ $y = 3x \sin x$ $\qquad y' = 3x \cos x + 3 \sin x$

$\boxed{c}$ $y = \dfrac{\sin x}{3x}$ $\qquad y' = \dfrac{3x \cos x - 3 \sin x}{9x^2}$

$\boxed{2}$ If $f(x) = \cos x$, then $f'(x) = -\sin x$

Proof:
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{\cos x (\cos h - 1) - \sin x \sin h}{h}$$

$$= \lim_{h\to 0}\left(\cos x \,\frac{\cosh-1}{h}\right) - \lim_{h\to 0}\left(\sin x \,\frac{\sin h}{h}\right)$$

$$= \cos x \lim_{h\to 0}\frac{\cos h-1}{h} - \sin x \lim_{h\to 0}\frac{\sin h}{h}$$

$$= \cos x \,(0) - \sin x \,(1) = -\sin x$$

Example: Find y' for the following functions:

[a] $y = 1-\sin x + \cos x \quad \Rightarrow y' = -\cos x - \sin x$

[b] $y = (1-\sin x)\cos x \quad \Rightarrow y' = \sin x(\sin x - 1) - \cos x \cos x$

$$= \sin^2 x - \sin x - \cos^2 x$$

[c] $y = \dfrac{\cos x}{1-\sin x} \quad \Rightarrow y' = \dfrac{-(1-\sin x)\sin x + \cos x \cos x}{(1-\sin x)^2}$

$$= \frac{\sin^2 x - \sin x + \cos^2 x}{(1-\sin x)^2} = \frac{1-\sin x}{(1-\sin x)^2} = \frac{1}{1-\sin x}$$

Simple Harmonic Motion: الحركة التوافقية البسيطة

Example: A weight hanging (معلق) from a spring is stretched down 3 units under its rest position and relased at time $t=0$. It's position at any time later on is

$$s = 3\cos t$$

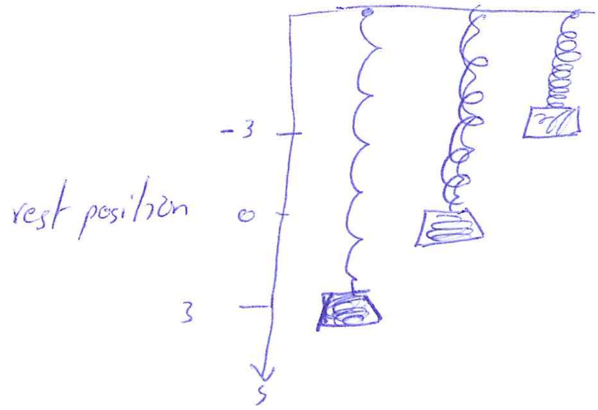
Find its velocity and acceleration, at time t?
jerk

Position: $s = 3\cos t$

Velocity $v = \dfrac{ds}{dt} = -3\sin t$

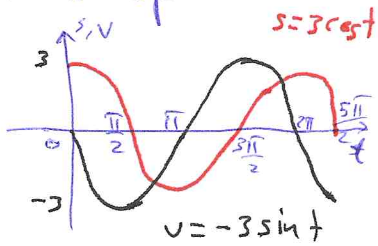
Acceleration $a = \dfrac{dv}{dt} = -3\cos t$

Jerk $j = \dfrac{da}{dt} = 3\sin t$



$\Rightarrow$

- As time passes, the weight moves down and up between $s = -3$ and $s = 3$ on s-axis.



- The amplitude of the motion is 3
- The period of the motion is 2π
- $v = -3\sin t$ gets greatest magnitude when $\cos t = 0$
- The speed $= |v| = 3|\sin t|$
- The acceleration is always the exact apposite of the position value. when weight is above the rest position, gravity is pulling it back down, when the weight is below the rest position, the spring is pulling it back up.

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3) If $f(x) = \tan x$, then $f'(x) = \sec^2 x$.

Proof $f(x) = \tan x = \dfrac{\sin x}{\cos x}$

$$f'(x) = \frac{\cos x \cos x + \sin x \sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

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4) If $f(x) = \cot x$, then $f'(x) = -\csc^2 x$

Proof $f(x) = \cot x = \dfrac{1}{\tan x}$

$$f'(x) = -\frac{\sec^2 x}{\tan^2 x} = -\frac{\frac{1}{\cancel{\cos^2 x}}}{\frac{\sin^2 x}{\cancel{\cos^2 x}}} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

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5) If $f(x) = \sec x$, then $f'(x) = \sec x \tan x$

Proof $f(x) = \dfrac{1}{\cos x}$

$$f'(x) = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$$

6 If $f(x) = \csc x$, then $f'(x) = -\csc x \cot x$

Proof: $f(x) = \dfrac{1}{\sin x}$

$$f'(x) = -\frac{\cos x}{\sin^2 x} = -\frac{1}{\sin x}\,\frac{\cos x}{\sin x} = -\csc x \cot x$$

Example: Let $y = \sec x \tan x$. Find y'

$$y' = \sec x \sec^2 x + \tan x \sec x \tan x$$

$$= \sec^3 x + \sec x \tan^2 x$$

Example: Find $\displaystyle\lim_{x\to 0} \frac{\sqrt{2+\sec x}}{\cos(\pi - \tan x)} = \frac{\sqrt{2+\sec 0}}{\cos(\pi - \tan 0)}$

$$= \frac{\sqrt{2+1}}{\cos(\pi - 0)} = \frac{\sqrt{3}}{\cos \pi} = -\sqrt{3}$$
