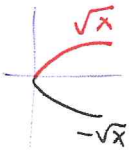
3.7 Implicit Differentiation

Example: Find $\frac{dy}{dx}$ if $y^2 = x$

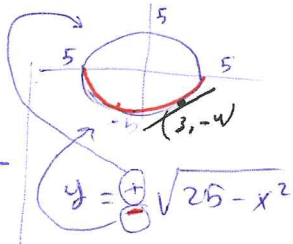
$$2yy' = 1 \Rightarrow y' = \frac{1}{2y}$$



Note that $y = \pm\sqrt{x} \Rightarrow y_1' = \frac{1}{2\sqrt{x}}$ for $y_1 = \sqrt{x}$

$y_2' = \frac{-1}{2\sqrt{x}}$ for $y_2 = -\sqrt{x}$

Example: Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.



slope: $2x + 2yy' = 0 \Rightarrow y' = \frac{-x}{y}$

slope at $(3,-4)$ is $y' = \frac{-3}{-4} = \frac{3}{4}$

Example: Find $\frac{dy}{dx}$ for $y^2 = x^2 + \sin xy$

$$2yy' = 2x + \cos xy\,(xy' + y)$$

$$2yy' - xy'\cos xy = 2x + y\cos xy$$

$$y'[2y - x\cos xy] = 2x + y\cos xy$$

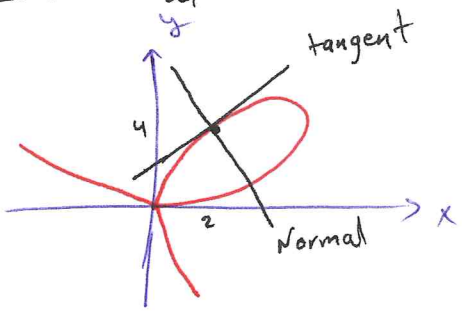
$$y' = \frac{2x + y\cos xy}{2y - x\cos xy}$$

Example: Find $\frac{d^2y}{dx^2}$ for $2x^3 - 3y^2 = 8$

$$6x^2 - 6yy' = 0 \Rightarrow y' = \frac{x^2}{y}, \quad y \neq 0$$

$$y'' = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2}y', \qquad y' = \frac{2x}{y} - \frac{x^4}{y^3}, \quad y \neq 0$$

Example: Find the tangent and the normal to the curve $x^3+y^3-9xy=0$ at the point $(2,4)$



$$3x^2+3y^2y'-9xy'-9y=0$$

$$y'(3y^2-9x)=9y-3x^2$$

$$y'=\frac{3y-x^2}{y^2-3x}$$

$$y'\Big|_{2,4}=\frac{3(4)-(2)^2}{(4)^2-3(2)}=\frac{12-4}{16-6}=\frac{8}{10}=\frac{4}{5}$$

- The tangent line at $(2,4)$ with slope $\frac{4}{5}$ is

$$y=\frac{4}{5}(x-2)+4=\frac{4}{5}x+\frac{12}{5}$$

- The normal line at $(2,4)$ is the line perpendicular to the tangent line at $(2,4)$

$$y=-\frac{5}{4}(x-2)+4=-\frac{5}{4}x+\frac{10}{4}+\frac{4}{1}\cdot\frac{4}{4}$$

$$y=-\frac{5}{4}x+\frac{13}{12}$$