$\boxed{3.8}$ Related Rates Equations

$\boxed{\uparrow = + \;,\; \downarrow = -}$

Suppose we are pumping (نفخ) air into a spherical balloon:
The volume (V) and radius (r) increase over time t:

$$V = \frac{4}{3}\pi r^3 \qquad \frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} \quad \text{(Chain Rule)}$$

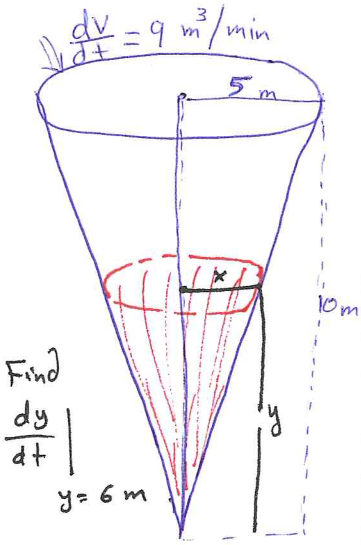
$\frac{m^3}{min}$ 8π $\quad$ 4 m $\quad$ 2

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

If we know r and $\frac{dV}{dt}$, then we can find $\frac{dr}{dt}$ ← how fast the radius increases over time

---

Example 1: Water runs into a conical tank at the rate $9\ m^3/min$. The tank stands point down and has a height 10 m and base radius 5 m. How fast is the water level rising when the water is 6 m deep?



The variables are: V, x, y For red with relation: $V = \frac{1}{3}\pi x^2 y$

We must eliminate x because we have no information about x and $\frac{dx}{dt}$. Similar triangles $\Rightarrow$

$$\frac{x}{y} = \frac{5}{10} \iff \boxed{x = \frac{y}{2}}$$

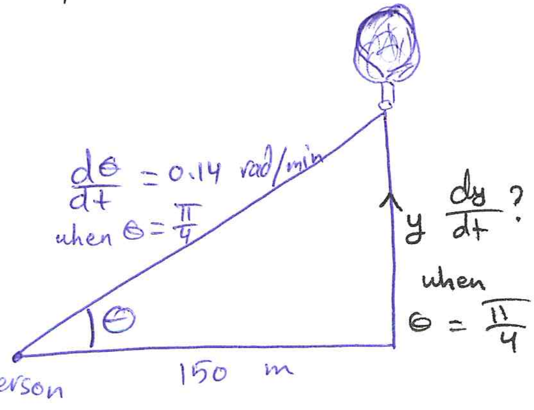
$$\Rightarrow V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y = \frac{\pi}{12} y^3$$

$$\frac{dV}{dt} = \frac{\pi}{4} y^2 \frac{dy}{dt} \iff 9 = \frac{\pi}{4}(6)^2 \frac{dy}{dt}$$

$$\iff \frac{dy}{dt} = \frac{1}{\pi} \approx 0.32\ m/min \quad \text{(the level is rising)}.$$

Example ② A balloon rising straight up from a level field (حقل مستوي) that is tracked (تعقّب) by a person whose is 150 m from the liftoff (الاقلاع) point. At the moment the person's elevation (زاوية الارتفاع) angle is $\frac{\pi}{4}$, the angle is increasing at rate of 0.14 rad/min. How fast is the balloon rising at that moment?

The Variables are θ and y with relation $\tan\theta = \frac{y}{150}$



$$y = 150\tan\theta$$

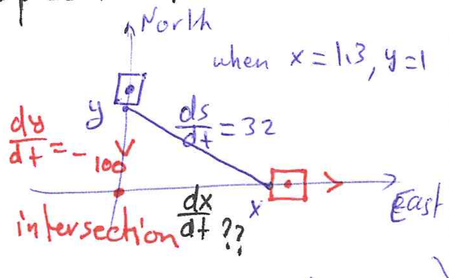
$$\frac{dy}{dt} = 150\sec^2\theta\,\frac{d\theta}{dt} \quad \Big|_{\theta=\frac{\pi}{4}}$$

$$= 150(\sqrt{2})^2(0.14)$$

$$= 42 \text{ m/min}$$ "the balloon is rising"

page 158

Example ③: A police car, approaching a right-angled intersection from the north, is chasing (مطاردة) a speeding car moving straight east. when the police is 1 Km north of the intersection and the car is 1.3 Km to the east, the police determine, using the radar, that the distance between them is increasing at rate 32 Km/hr. If the police is moving at 100 Km/hr at the instance of measurament, what is the speed of the car?



The variables are x, y, s and related by $s^2 = x^2 + y^2$

$$\cancel{2}s\frac{ds}{dt} = \cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt}$$

$$\sqrt{x^2+y^2}\,\frac{ds}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

$$\Leftrightarrow \sqrt{(1.3)^2 + (1)^2}\,(32) = (1.3)\frac{dx}{dt} + (1)(-100)$$

$$\Leftrightarrow \frac{dx}{dt} = 117.3 \text{ Km/hr}$$

car's speed.