$\boxed{3.9}$ linearization and Differentials

Def: If f is differentiable at $x = a$, then the approximating function
$$L(x) = f'(a)(x-a) + f(a)$$ is the linearization of f at a.

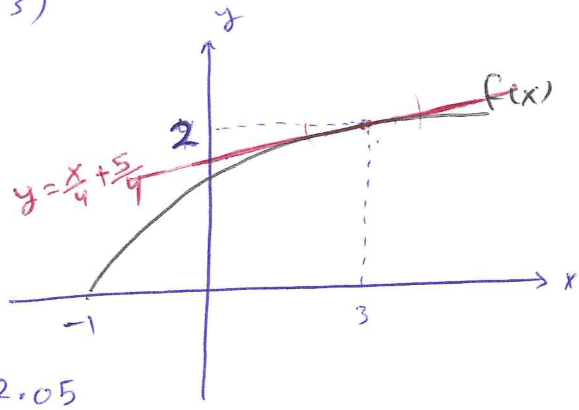
- We approximate f by L and we write $f(x) \approx L(x)$ is the standard linear approximation of f at a.
- The point $x = a$ is the center of the approximation.

Example: Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 3$.

$f(3) = \sqrt{1+3} = \sqrt{4} = 2$, $\quad f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$

$f'(3) = \frac{1}{2} \frac{1}{\sqrt{1+3}} = \frac{1}{4}$

$$f(x) \approx L(x) = f'(3)(x-3) + f(3)$$
$$= \frac{1}{4}(x-3) + 2$$
$$= \frac{x}{4} + \frac{5}{4}$$



Take $x = 3.2 \Rightarrow L(3.2) = \frac{3.2}{4} + \frac{5}{4}$
$= \frac{8.2}{4} = 2.05$

$\Rightarrow f(3.2) = \sqrt{1+3.2} = \sqrt{4.2} \approx 2.04939$

Example: Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 0$.

$f(0) = \sqrt{1+0} = 1$, $f'(x) = \dfrac{1}{2\sqrt{1+x}}$

$f'(0) = \frac{1}{2}$

$L(x) = f'(0)(x-0) + f(0)$

$L(x) = \frac{x}{2} + 1$

- If $f(x) = (1+x)^k$, k any number, then the linearization of f at $x=0$ is $L(x) = 1 + kx$
- f can be any roots or powers

Example: $(1+x)^{\frac{1}{2}} \approx 1 + \frac{x}{2}$

$$\frac{1}{1-x} = (1-x)^{-1} \approx 1 + x$$

$$\sqrt[3]{1+5x^4} = (1+5x^4)^{\frac{1}{3}} \approx 1 + \frac{1}{3}(5x^4) = 1 + \frac{5}{3}x^4$$

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} \approx 1 + \left(-\tfrac{1}{2}\right)(-x^2) = 1 + \frac{x^2}{2}$$

Example: Estimate $(1.0001)^{1000}$

$$(1.0001)^{1000} = (1 + 0.0001)^{1000} \approx 1 + (0.0001)(1000) = 1 + 0.1 = 1.1$$

Estimate $\sqrt{1.004}$

$$(1.004)^{\frac{1}{2}} = (1 + 0.004)^{\frac{1}{2}} \approx 1 + 0.004\left(\tfrac{1}{2}\right) = 1 + 0.002 = 1.002$$

Def: Let $y = f(x)$ be differentiable function.
The differential dy is

$$dy = f'(x)\, dx$$ where dx is the independent differential

Example: Find dy if $y = x^3 - 3\sqrt{x}$

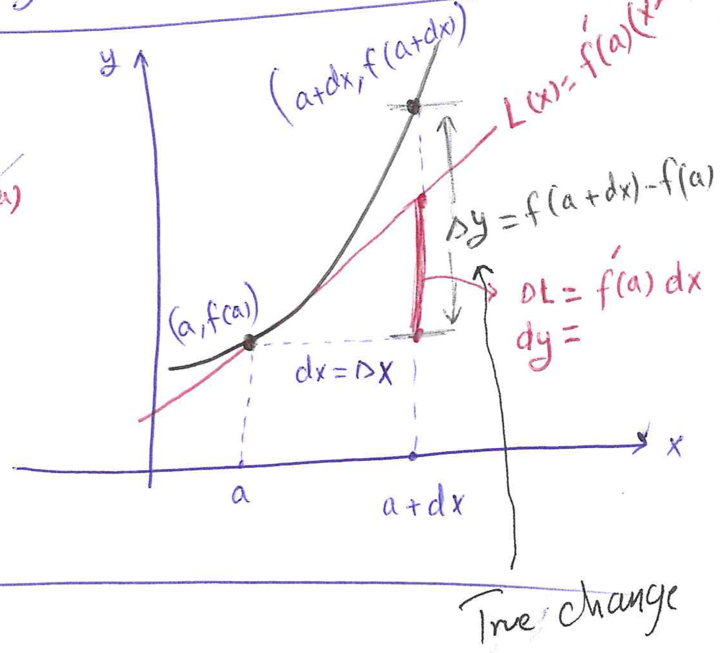
$$dy = 3x^2\, dx - \frac{3}{2} x^{-\frac{1}{2}}\, dx$$
$$= 3\left[x^2 - \frac{1}{2\sqrt{x}}\right] dx$$

Find the differential dy if

$$xy^2 - 4x^{3/2} - y = 0$$
$$y^2\, dx + 2yx\, dy - 6x^{\frac{1}{2}}\, dx - dy = 0$$
$$dy\,[2yx - 1] = \left(6x^{\frac{1}{2}} - y^2\right) dx$$
$$dy = \frac{6\sqrt{x} - y^2}{2xy - 1}\, dx$$

Estimated change

$$\Delta L = L(a + dx) - L(a)$$
$$= f'(a)\left((\cancel{a} + dx) - \cancel{a}\right) + \cancel{f(a)} - \cancel{f(a)}$$
$$= f'(a)\, dx$$
$$= dy$$



Estimating with Differentials

True change $\Delta f = f(a + \underset{\Delta x}{dx}) - f(a)$

Estimated change $df = f'(a)\, dx$ ($dx = \Delta x$)

↳ Differential

Relative True change $\dfrac{\Delta f}{f(a)}$

Relative Estimated change $\dfrac{df}{f(a)}$

True Percentage change $\dfrac{\Delta f}{f(a)} \times 100$

Estimated Percentage change $\dfrac{df}{f(a)} \times 100$

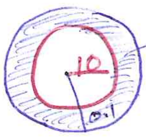
Sensitivity to change

$df = f'(x)\, dx$

how sensitive the output f is ~~to~~ a change in the input at different values x

Example: The radius of a circle increases from $a = 10$ m to 10.1 m.

[a] Estimate the increase in the circle's area.



$\Rightarrow A(r) = r^2 \pi$

$A' = 2r\pi$

$A'(10) = 2(10)\pi = 20\pi$

$dA = A'(10)\, dr$

$= (20\pi)(0.1)$

$= 2\pi \ m^2$

$\Rightarrow \underset{\Delta r}{dr} = r_2 - r_1 = 10.1 - 10 = 0.1$ m

[d] Estimate the enlarged circle area and compar it with the true area

• $A(10.1) \approx A(10) + dA$

$(10)^2\pi + 2\pi = 102\pi \ m^2$

• True area $A(10.1) = (10.1)^2 \pi = 102.01\pi \ m^2$

[b] Find the true change in the area?

$\Delta A = A(10.1) - A(10) = (102.01 - 100)\pi = 2.01\pi \ m^2$

[c] Find the error? $\left| \underset{2.01\pi}{\Delta A} - \underset{2\pi}{dA} \right| = 0.01\pi \ m^2 = \epsilon \Delta x$

Error in Differential Approximation

Approximating error = The true change − The differential Estimated change

$$= \Delta f - df$$

$$= f(a+\Delta x) - f(a) - f'(a)\,\Delta x$$

$$= \left[\frac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a)\right]\Delta x$$

$$= \epsilon \cdot \Delta x$$

as $\Delta x \to 0 \Rightarrow$. $\frac{f(a+\Delta x) - f(a)}{\Delta x} \to f'(a)$

thus, $\epsilon \to 0$ which is very small.

$$\underbrace{\Delta f}_{\text{True change}} = \underbrace{df}_{\text{Estimated change}} + \underbrace{\epsilon\,\Delta x}_{\text{Error}}$$

$$\boxed{\Delta f = f'(a)\,\Delta x + \epsilon\,\Delta x}$$

In the previous example $\Rightarrow$ $\Delta A = 2.01\,\pi\,m^2$

$dA = 2\pi \ m^2$

$\Delta r = 0.1\ m$

$$\Delta A = dA + \epsilon\,\Delta r$$

$2.01\pi = 2\pi + \epsilon\ 0.1 \Leftrightarrow \epsilon\ 0.1 = 0.01\pi \Leftrightarrow \epsilon = 0.1\pi\ m$

approximating error

Example: How does a 10% decrease in r affect V if $V = kr^4$.

$dV = 4kr^3\,dr \Rightarrow$ The Estimated relative change $\frac{dV}{V} = \frac{4kr^3}{kr^4}\,dr$

$$\Leftrightarrow \frac{dV}{V} = 4\,\frac{dr}{r}$$

The relative change in V is 4 times the relative change in r.

Thus, a decrease of 10% r will decrease V by 40%.