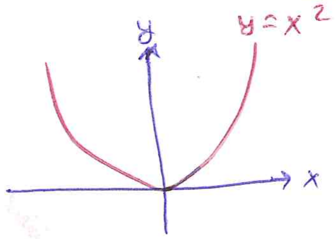
$\boxed{4.1}$ Extreme Values of Functions

Def: Let f be a function with domain D.

- f has an absolute (or global) maximum value on D at c if
$$f(c) \geq f(x) \quad \text{for all } x \in D.$$
- f has an absolute (or global) minimum value on D at c if
$$f(c) \leq f(x) \quad \text{for all } x \in D.$$

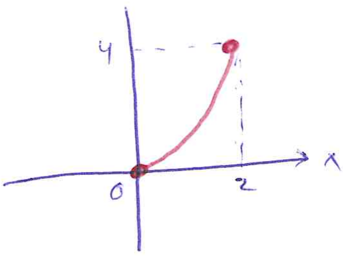
- Maximum and minimum values are also called extreme values.
- The function **might** not have a maximum or minimum if the domain is unbounded or is not a closed interval.

Example $\boxed{a}$ $y = x^2$ on $(-\infty, \infty)$



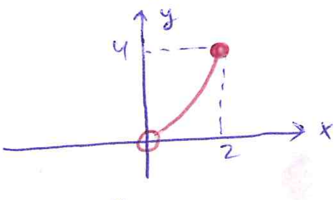
No absolute maximum
Absolute minimum at $(0,0)$
(of 0 at $x=0$)

$\boxed{b}$ $y = x^2$ on $[0,2]$



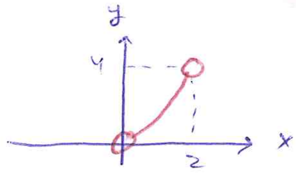
Absolute maximum of 4 at $x = 2$ $(2,4)$
Absolute minimum of 0 at $x = 0$ $(0,0)$

$\boxed{c}$ $y = x^2$ on $(0,2]$



Absolute maximum of 4 at $x=2$
No absolute minimum

$\boxed{d}$ $y = x^2$ on $(0,2)$

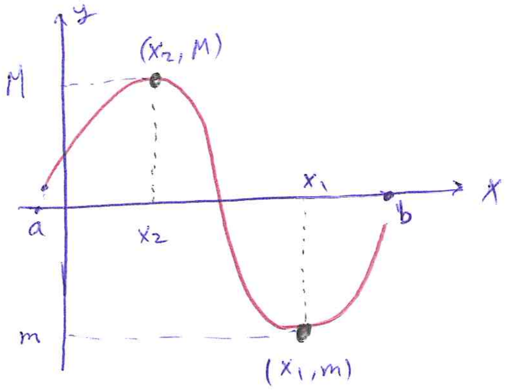


No absolute max
No absolute min

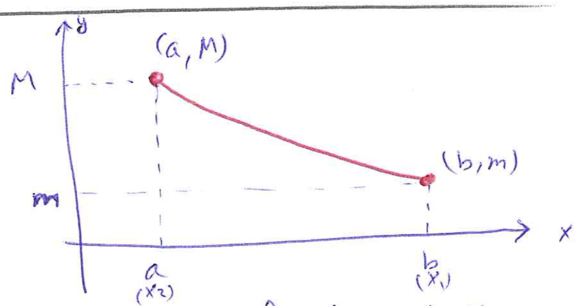
Theorem 1 (The Extreme Value Theorem)

If f is continuous on a closed interval $[a,b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a,b]$.

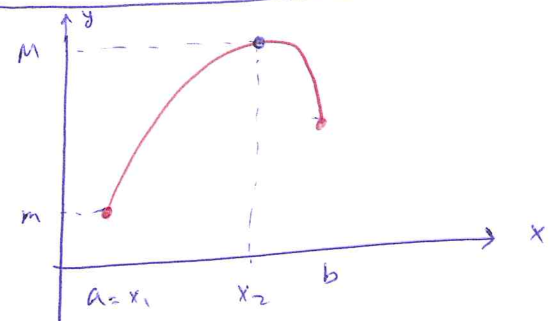
That is, there are numbers x_1 and $x_2 \in [a,b]$ with $f(x_1) = m$ and $f(x_2) = M$ and $m \le f(x) \le M$ for every other x in $[a,b]$



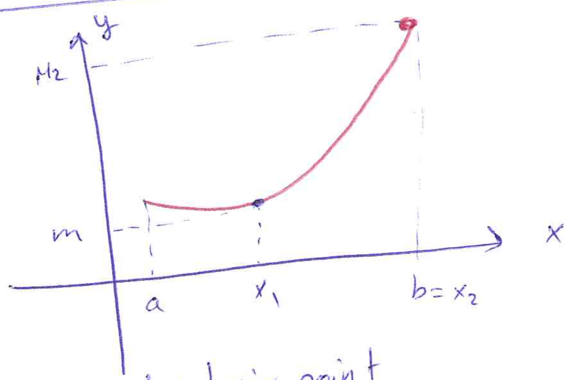
Max and min are interior points



Max and min at the end points

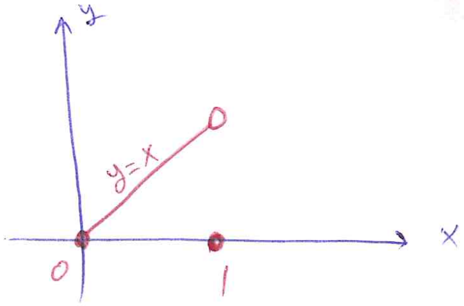


Max at interior point
min at endpoint



Min at interior point
max at end point

Example: $f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$

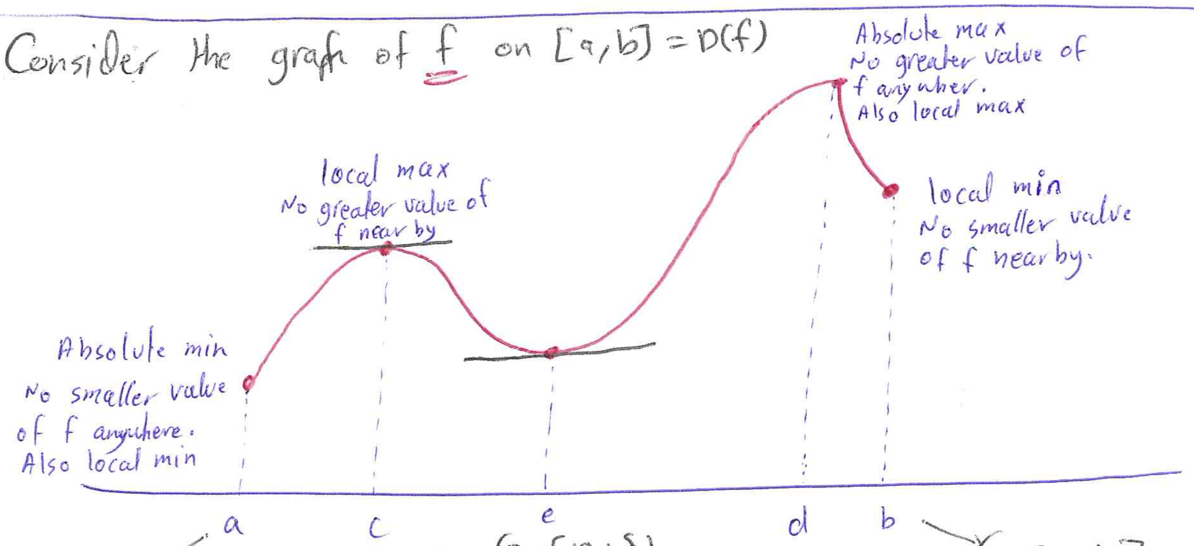


- Absolute minimum at $(0,0)$ and $(1,0)$ of 0 at $x = 0, 1$
- No absolute max because of discontinuity at 1.

Def • A function f has a local (or Relative) maximum at point $c \in D(f)$ if $f(c) \geq f(x)$ for all $x \in D$ lying in some open interval contains c

• A function f has a local (or Relative) minimum at point $c \in D(f)$ if $f(c) \leq f(x)$ for all $x \in D$ lying in some open interval contains c.

Consider the graph of f on $[a,b] = D(f)$



$[a, a+\delta)$ half-open interval; $(c-\delta, c+\delta)$ open interval; $(e-\delta, e+\delta)$ open interval; $(d-\delta, d+\delta)$; $(b-\delta, b]$ Half open interval

• f has local max at c and d (d is
• f has local min at a, e and b

• local extrema are also called relative extrema.

• Absolute max is also local max
• Absolute min is also local min

Theorem: If f has a local maximum or minimum value at an interior point $c \in D(f)$ and if $f'(c)$ is defined then $f'(c) = 0$

Proof: Suppose that f has a local max (for example) at $x=c$
$\Rightarrow f(c) \geq f(x)$ for all x near c
$\Rightarrow$ Since c is an interior point and $f'(c)$ is defined

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0 \quad \text{and} \quad f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0 \quad \Leftrightarrow \quad f'(c) = 0$$

Def: An interior point $c \in D(f)$ where $f'(c) = 0$ or $f'(c)$ is undefined is called a critical point.

Example: Find the critical points of $y = x^2 - 32\sqrt{x}$

$y' = 2x - \dfrac{16}{\sqrt{x}} = 0 \iff x - \dfrac{8}{\sqrt{x}} = 0$

$\iff \dfrac{x^{\frac{3}{2}} - 8}{\sqrt{x}} = 0 \iff x^{\frac{3}{2}} - 8 = 0 \iff \left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} = \left(2^{3}\right)^{\frac{2}{3}} \iff \boxed{x = 4}$

$\boxed{x = 0}$

To find the Absolute Exterema of a continuous function f on $[a, b]$:

1. Evaluate f at all critical points and endpoint.
2. Take the largest and smallest of these values.

Example: Find absolute maximum and minimum of

① $f(x) = \frac{2}{3}x - 5 \qquad -3 \le x \le 6$

$f'(x) = \frac{2}{3} \ne 0$ no critical points

$f(-3) = \frac{2}{3}(-3) - 5 = -2 - 5 = -7$

$f(6) = \frac{2}{3}(6) - 5 = 4 - 5 = -1$

f has absolute max of -1 at $x = 6$

f has absolute min of -7 at $x = -3$

② $g(x) = x^2 - 32\sqrt{x}$ on $[1, 9]$

critical points ~~are $x = 0$ and~~ is $x = 4$ since $x = 0 \notin D(g)$

~~$g(0) = 0$ (Absolute max of 0 at $x=0$).~~

✓ $g(4) = 16 - 32(2) = 16 - 64 = -48$ (Absolut min at $(4, -48)$)

$g(1) = 1 - 32 = -31$

$g(9) = 81 - (32)(3) = 81 - 96 = -15$ (Absolut max of -15 at $x = 9$) ✓